ASE2010 Applied linear algebra: Homework #5

1) *Quadratic form.* Suppose P is an $n \times n$ matrix. The function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = x^T P x$ is called a *quadratic form*, and generalizes the idea of a quadratic function of a scalar variable, px^2. The matrix P is called the coefficient matrix of the quadratic form.

   a) Show that $f(x) = \sum_{i,j} P_{ij} x_i x_j$. In words: $f(x)$ is the weighted sum of all products of two components of x, with weights given by the entries of P.

   b) Show that for any x, we also have $f(x) = x^T P^T x$. In other words, the quadratic form associated with the transpose matrix is the same function.

   c) Show that f can be expressed as $f(x) = x^T P^s x$, where $P^s = (1/2)(P + P^T)$ is the symmetric part of P. The matrix P^s is symmetric. So any quadratic form can be expressed as one with a coefficient matrix that is symmetric.

   d) Express $f(x) = -2x_1^2 + 4x_1x_2 + 2x^2$ in the form $f(x) = x^T P x$ with P a symmetric 2×2 matrix.

   e) Suppose that A is an $m \times n$ matrix and b is an m-vector. Show that $\|Ax - b\|^2 = x^T P x + q^T x + r$ for a suitable $n \times n$ symmetric matrix P, n-vector q, and constant r. (Give P , q, and r.) In words: The norm squared of an affine function of x can be expressed as the sum of a quadratic form and an affine function.

2) *Matrix identities.* Check that the following identities regarding matrix inverses hold. You can assume that X,Y,Z are matrices in appropriate sizes, and a, b are vectors in appropriate sizes. You can also assume that the appearing inverses exist.

   a)
   $$Z(I + Z)^{-1} = I - (I + Z)^{-1}$$

   b)
   $$(I + XY)^{-1} = I - X(I + YX)^{-1}Y$$

   c)
   $$Y(I + XY)^{-1} = (I + YX)^{-1}Y$$

   d)
   $$(I + XZ^{-1}Y)^{-1} = I - X(Z + YX)^{-1}Y$$

   e)
   $$(X + ab^T)^{-1} = X^{-1} - \frac{1}{1 + b^T X^{-1} a} X^{-1} a b^T X^{-1}$$

3) *VMLS Exercises.*

   a) **11.6** *Inverse of a block upper triangular matrix.*

   b) **11.12** *Combinations of invertible matrices.*

   c) **11.14** *Middle inverse.*

   d) **11.15** *Invertibility of population dynamics matrix.*