17. Constrained least squares applications

Outline

Portfolio optimization

Linear quadratic control

Linear quadratic state estimation

Portfolio allocation weights

- we invest a total of V dollars in n different *assets* (stocks, bonds, ...) over some period (one day, week, month, ...)
- can include *short positions*, assets you borrow and sell at the beginning, but must return to the borrower at the end of the period
- *portfolio allocation weight vector* w gives the fraction of our total portfolio value held in each asset
- Vw_j is the dollar value of asset j you hold
- $\mathbf{1}^T w = 1$, with negative w_i meaning a short position
- $w = (-0.2, 0.0, 1.2)$ means we take a short position of $0.2V$ in asset 1, don't hold any of asset 2, and hold $1.2V$ in asset 3

Leverage, long-only portfolios, and cash

- *leverage* is $L = |w_1| + \cdots + |w_n|$
  ($(L-1)/2$ is also sometimes used)
- $L = 1$ when all weights are nonnegative ('long only portfolio')
- $w = \mathbf{1}/n$ is called the *uniform portfolio*

- we often assume asset n is 'risk-free' (or cash or T-bills)
- so $w = e_n$ means the portfolio is all cash

Return over a period

- $\tilde{r}_j$ is the *return* of asset j over the period
- $\tilde{r}_j$ is the fractional increase in price or value (decrease if negative)
- often expressed as a percentage, like $+1.1\%$ or -2.3%
- full *portfolio return* is

$$\frac{V^+ - V}{V} = \tilde{r}^T w$$

  where V^+ is the portfolio value at the end of the period
- if you hold portfolio for t periods with returns $r_1, \ldots, r_t$ value is

$$V_{t+1} = V_1(1 + r_1)(1 + r_2) \cdots (1 + r_t)$$

- portfolio value versus time traditionally plotted using $V_1 = \$10000$

Return matrix

- hold portfolio with weights w over T periods
- define $T \times n$ (asset) *return matrix*, with R_{tj} the return of asset j in period t
- row t of R is $\tilde{r}_t^T$, where $\tilde{r}_t$ is the asset return vector over period t
- column j of R is time series of asset j returns
- portfolio returns vector (time series) is T-vector $r = Rw$
- if last asset is risk-free, the last column of R is $\mu^{\mathrm{rf}}\mathbf{1}$, where μ^{rf} is the risk-free per-period interest rate

Portfolio return and risk

- r is time series (vector) of portfolio returns
- *average return* or just *return* is $\mathbf{avg}(r)$
- *risk* is $\mathbf{std}(r)$
- these are the per-period return and risk

- for small per-period returns we have

$$
\begin{aligned}
V_{T+1} &= V_1(1+r_1)\cdots(1+r_T) \\
&\approx V_1 + V_1(r_1+\cdots+r_T) \\
&= V_1 + T\,\mathbf{avg}(r)V_1
\end{aligned}
$$

- so return approximates the average per-period increase in portfolio value

Annualized return and risk

- mean return and risk are often expressed in *annualized form* (*i.e.*, per year)
- if there are P trading periods per year

$$\text{annualized return} = P\,\mathbf{avg}(r), \qquad \text{annualized risk} = \sqrt{P}\,\mathbf{std}(r)$$

  (the squareroot in risk annualization comes from the assumption that the fluctuations in return around the mean are independent)

- if returns are daily, with 250 trading days in a year

$$\text{annualized return} = 250\,\mathbf{avg}(r), \qquad \text{annualized risk} = \sqrt{250}\,\mathbf{std}(r)$$

Portfolio optimization

- how should we choose the portfolio weight vector w?
- we want high (mean) portfolio return, low portfolio risk

- we know past *realized asset returns* but not future ones
- we will choose w that would have worked well on past returns
- ... and hope it will work well going forward (just like data fitting)

Portfolio optimization

$$
\begin{array}{ll}
\text{minimize} & \mathbf{std}(Rw)^2 = (1/T)\|Rw - \rho\mathbf{1}\|^2 \\
\text{subject to} & \mathbf{1}^T w = 1 \\
& \mathbf{avg}(Rw) = \rho
\end{array}
$$

- w is the weight vector we seek
- R is the returns matrix for *past returns*
- Rw is the (past) portfolio return time series
- require mean (past) return ρ
- we minimize risk for specified value of return
- solutions w are *Pareto optimal*

- we are really asking what *would have been* the best constant allocation, had we known future returns

Portfolio optimization via constrained least squares

$$\begin{array}{ll} \text{minimize} & \|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \left[\begin{array}{c} \mathbf{1}^T \\ \mu^T \end{array}\right] w = \left[\begin{array}{c} 1 \\ \rho \end{array}\right] \end{array}$$

- $\mu = R^T\mathbf{1}/T$ is n-vector of (past) asset returns
- ρ is required (past) portfolio return
- an equality constrained least squares problem, with solution

$$\left[\begin{array}{c} w \\ z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{ccc} 2R^TR & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 2\rho T\mu \\ 1 \\ \rho \end{array}\right]$$

Optimal portfolios

- perform significantly better than individual assets
- risk-return curve forms a straight line
- one end of the line is the risk-free asset
- *two-fund theorem:* optimal portfolio w is an affine function of ρ

$$\left[\begin{array}{c} w \\ z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{ccc} 2R^TR & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] + \rho \left[\begin{array}{ccc} 2R^TR & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 2T\mu \\ 0 \\ 1 \end{array}\right]$$

The big assumption

- now we make the big assumption (BA):

  FUTURE RETURNS WILL LOOK SOMETHING LIKE PAST ONES

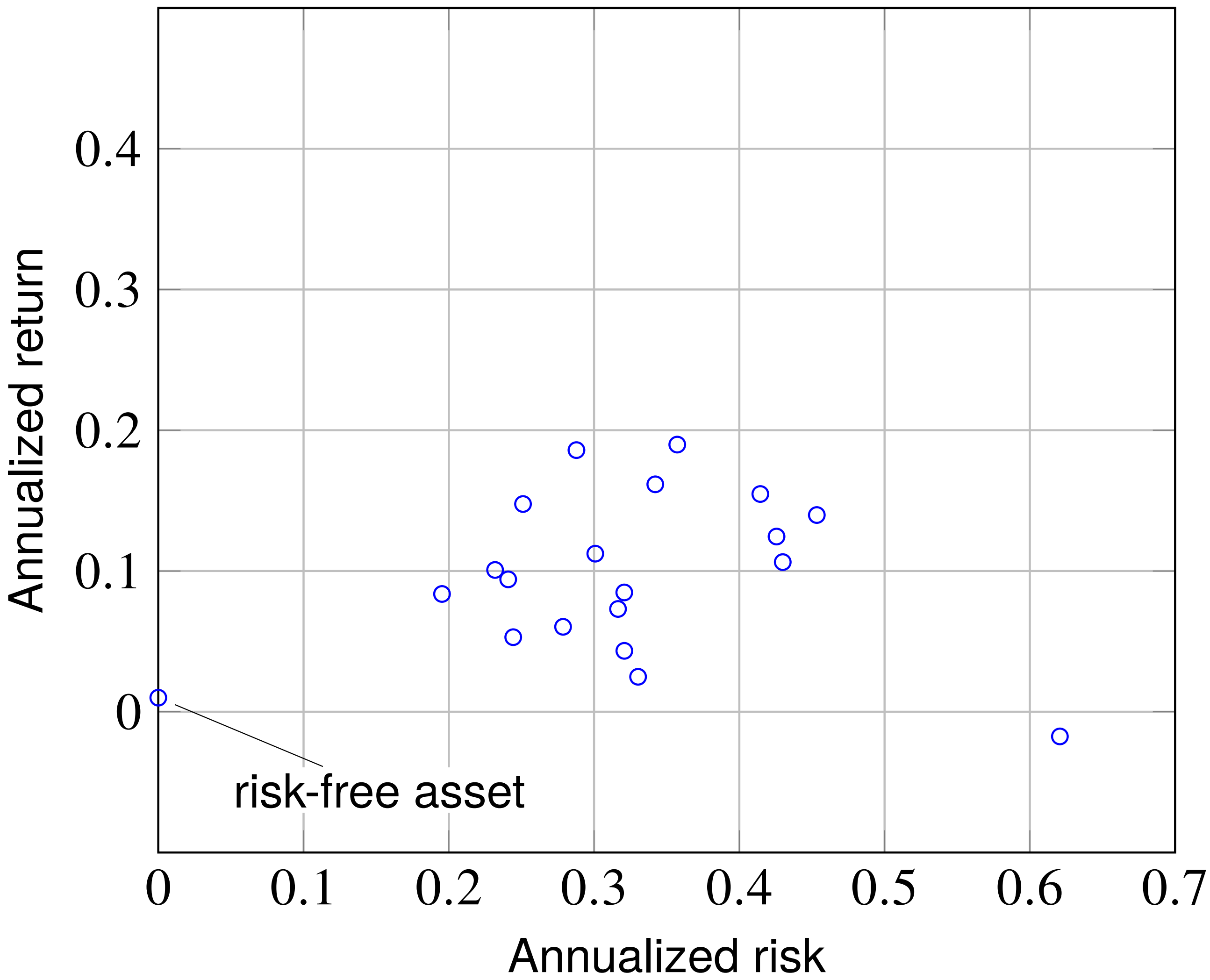
  - you are warned this is false, every time you invest
  - it is often reasonably true
  - in periods of 'market shift' it's much less true

- if BA holds (even approximately), then a good weight vector for past (realized) returns should be good for future (unknown) returns

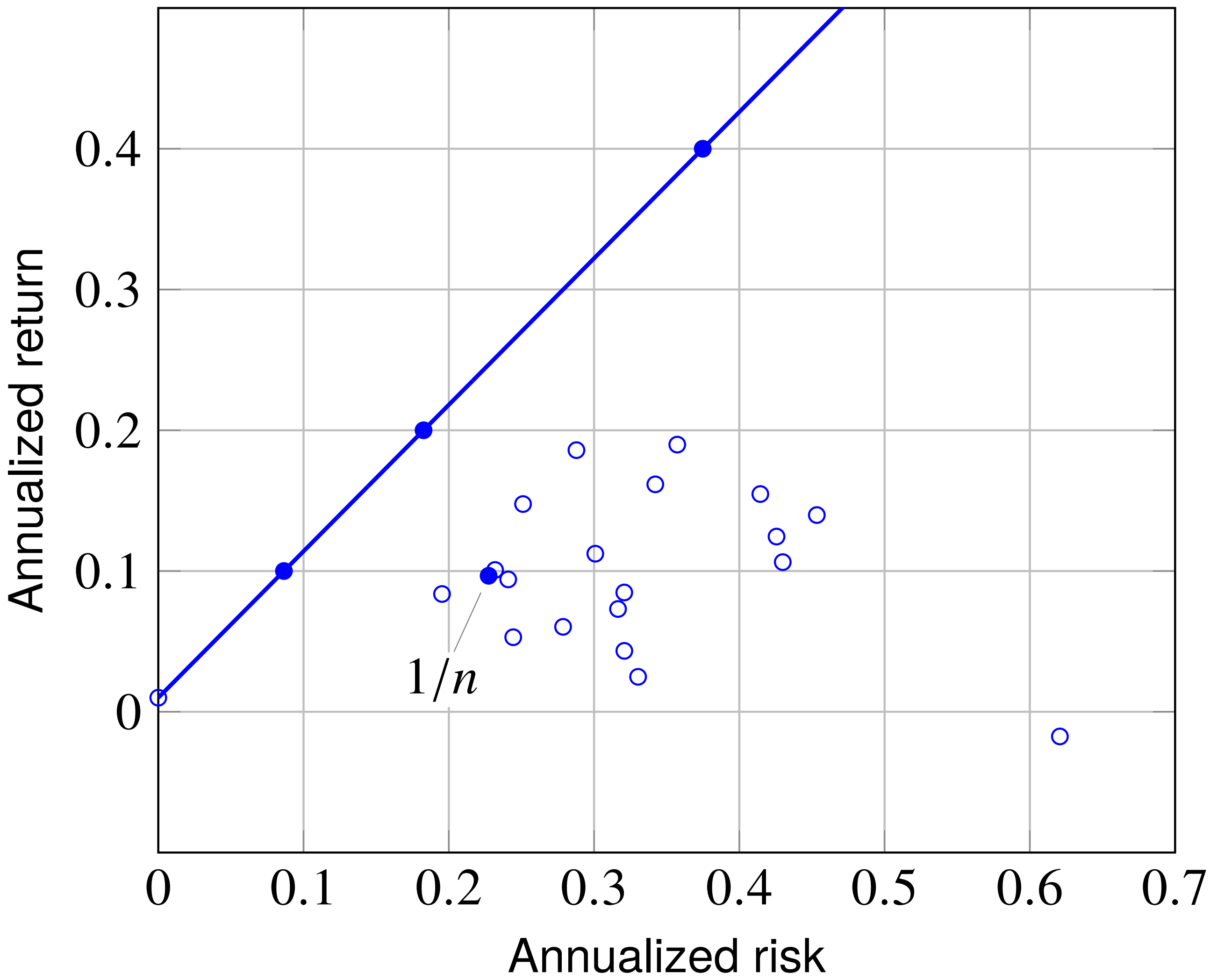
- for example:
  - choose w based on last 2 years of returns
  - then use w for next 6 months

Example

20 assets over 2000 days



Pareto optimal portfolios

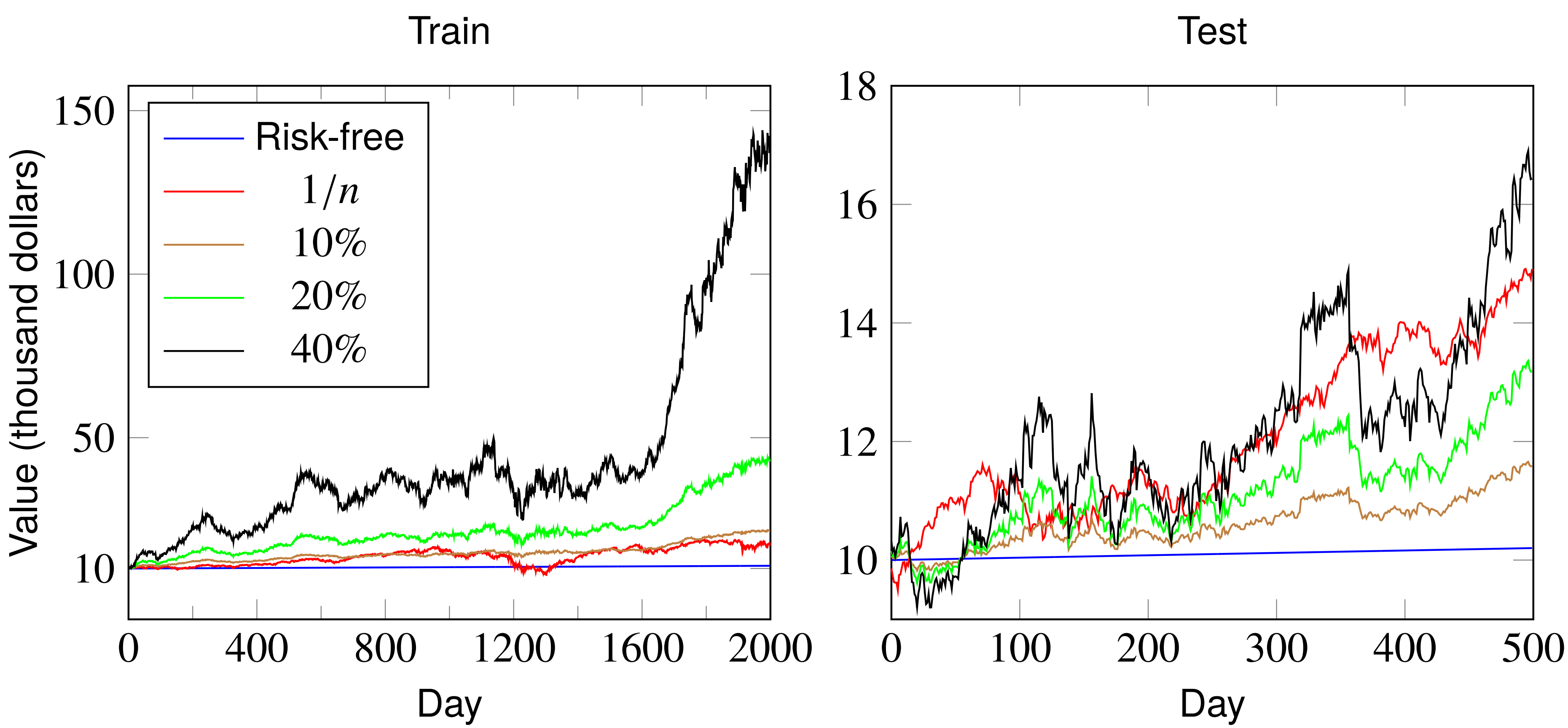


Five portfolios

	Return		Risk		
Portfolio	Train	Test	Train	Test	Leverage
risk-free	0.01	0.01	0.00	0.00	1.00
$\rho = 10\%$	0.10	0.08	0.09	0.07	1.96
$\rho = 20\%$	0.20	0.15	0.18	0.15	3.03
$\rho = 40\%$	0.40	0.30	0.38	0.31	5.48
$1/n$ (uniform weights)	0.10	0.21	0.23	0.13	1.00

- train period of 2000 days used to compute optimal portfolio
- test period is different 500-day period

Total portfolio value



Outline

Portfolio optimization

Linear quadratic control

Linear quadratic state estimation

Linear dynamical system

$$x_{t+1} = A_t x_t + B_t u_t, \quad y_t = C_t x_t, \quad t = 1, 2, \ldots$$

- n-vector x_t is *state* at time t
- m-vector u_t is *input* at time t
- p-vector y_t is *output* at time t
- $n \times n$ matrix A_t is *dynamics matrix*
- $n \times m$ matrix B_t is *input matrix*
- $p \times n$ matrix C_t is *output matrix*
- x_t, u_t, y_t often represent deviations from a standard operating condition

Linear quadratic control

$$
\begin{array}{ll}
\text{minimize} & J_{\text{output}} + \rho J_{\text{input}} \\
\text{subject to} & x_{t+1} = A_t x_t + B_t u_t, \quad t = 1, \ldots, T-1 \\
& x_1 = x^{\text{init}}, \quad x_T = x^{\text{des}}
\end{array}
$$

- variables are state sequence $x_1, \ldots, x_T$ and input sequence $u_1, \ldots, u_{T-1}$
- two objectives are quadratic functions of state and input sequences:

$$
\begin{array}{rcl}
J_{\text{output}} & = & \|y_1\|^2 + \cdots + \|y_T\|^2 \;=\; \|C_1 x_1\|^2 + \cdots + \|C_T x_T\|^2 \\
J_{\text{input}} & = & \|u_1\|^2 + \cdots + \|u_{T-1}\|^2
\end{array}
$$

- first constraint imposes the linear dynamics equations
- second set of constraints specifies the initial and final state
- ρ is positive parameter used to trade off the two objectives

Constrained least squares formulation

$$
\begin{array}{ll}
\text{minimize} & \|C_1x_1\|^2 + \cdots + \|C_Tx_T\|^2 + \rho\|u_1\|^2 + \cdots + \rho\|u_{T-1}\|^2 \\
\text{subject to} & x_{t+1} = A_tx_t + B_tu_t, \quad t = 1, \ldots, T-1 \\
& x_1 = x^{\text{init}}, \quad x_T = x^{\text{des}}
\end{array}
$$

- can be written as

$$
\begin{array}{ll}
\text{minimize} & \|\tilde{A}z - \tilde{b}\|^2 \\
\text{subject to} & \tilde{C}z = \tilde{d}
\end{array}
$$

- vector z contains the $Tn + (T-1)m$ variables:

$$
z = (x_1, \ldots, x_T, u_1, \ldots, u_{T-1})
$$

Constrained least squares formulation

$$\tilde{A} = \left[\begin{array}{ccc|ccc} C_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & C_T & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \sqrt{\rho}I & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \sqrt{\rho}I \end{array}\right], \qquad \tilde{b} = 0$$

$$\tilde{C} = \left[\begin{array}{cccccc|cccc} A_1 & -I & 0 & \cdots & 0 & 0 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & -I & \cdots & 0 & 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{T-1} & -I & 0 & 0 & \cdots & B_{T-1} \\ \hline I & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 & 0 & \cdots & 0 \end{array}\right], \qquad \tilde{d} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline x^{\text{init}} \\ x^{\text{des}} \end{array}\right]$$

Example

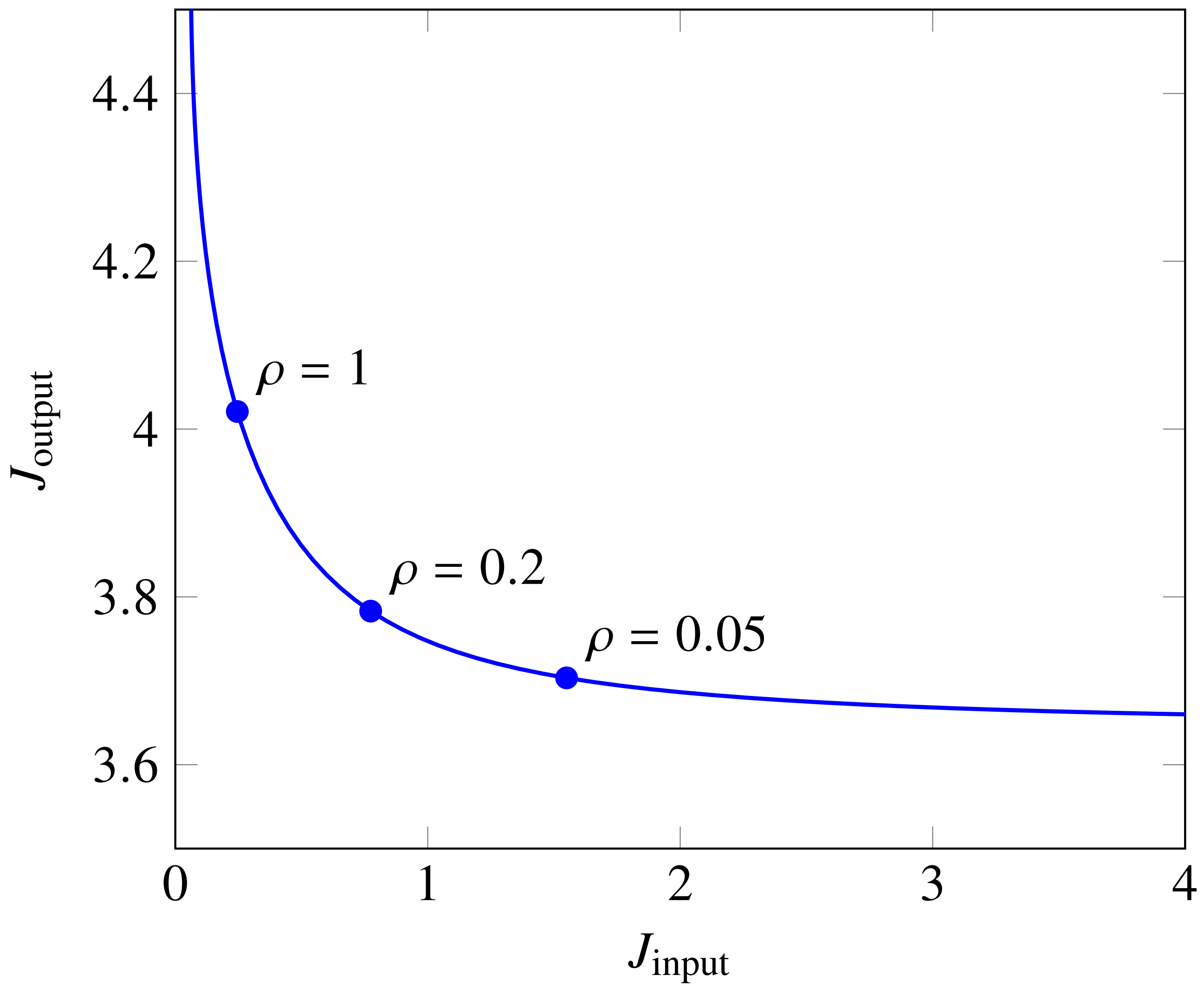
- time-invariant system: system matrices are constant

$$A = \begin{bmatrix} 0.855 & 1.161 & 0.667 \\ 0.015 & 1.073 & 0.053 \\ -0.084 & 0.059 & 1.022 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.076 \\ -0.139 \\ 0.342 \end{bmatrix},$$

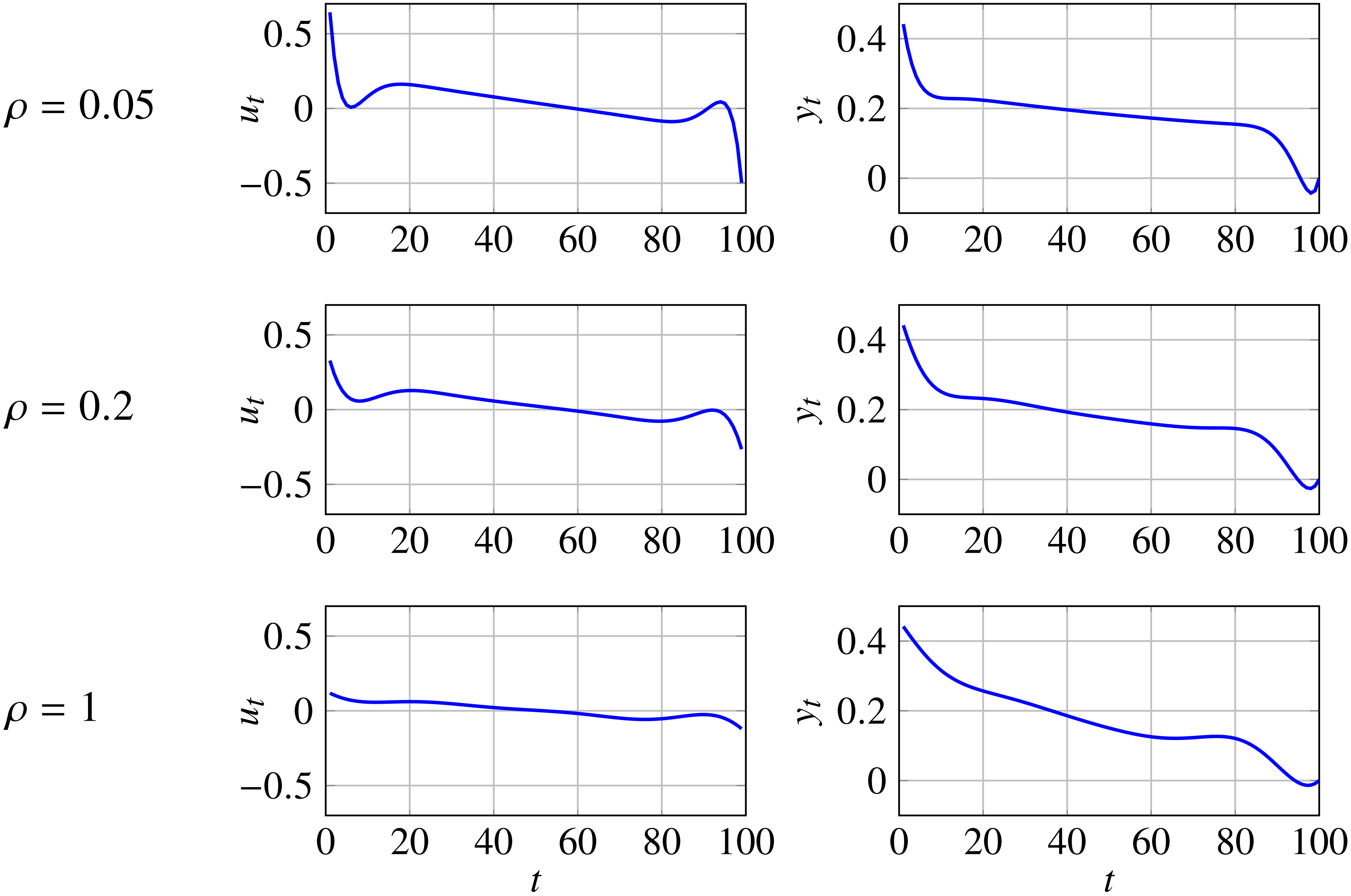
$$C = \begin{bmatrix} 0.218 & -3.597 & -1.683 \end{bmatrix}$$

- initial condition $x^{\text{init}} = (0.496, -0.745, 1.394)$
- target or desired final state $x^{\text{des}} = 0$
- $T = 100$

Optimal trade-off curve



Three points on the trade-off curve



Linear state feedback control

- linear state feedback control uses the input

$$u_t = Kx_t, \quad t = 1, 2, \ldots$$

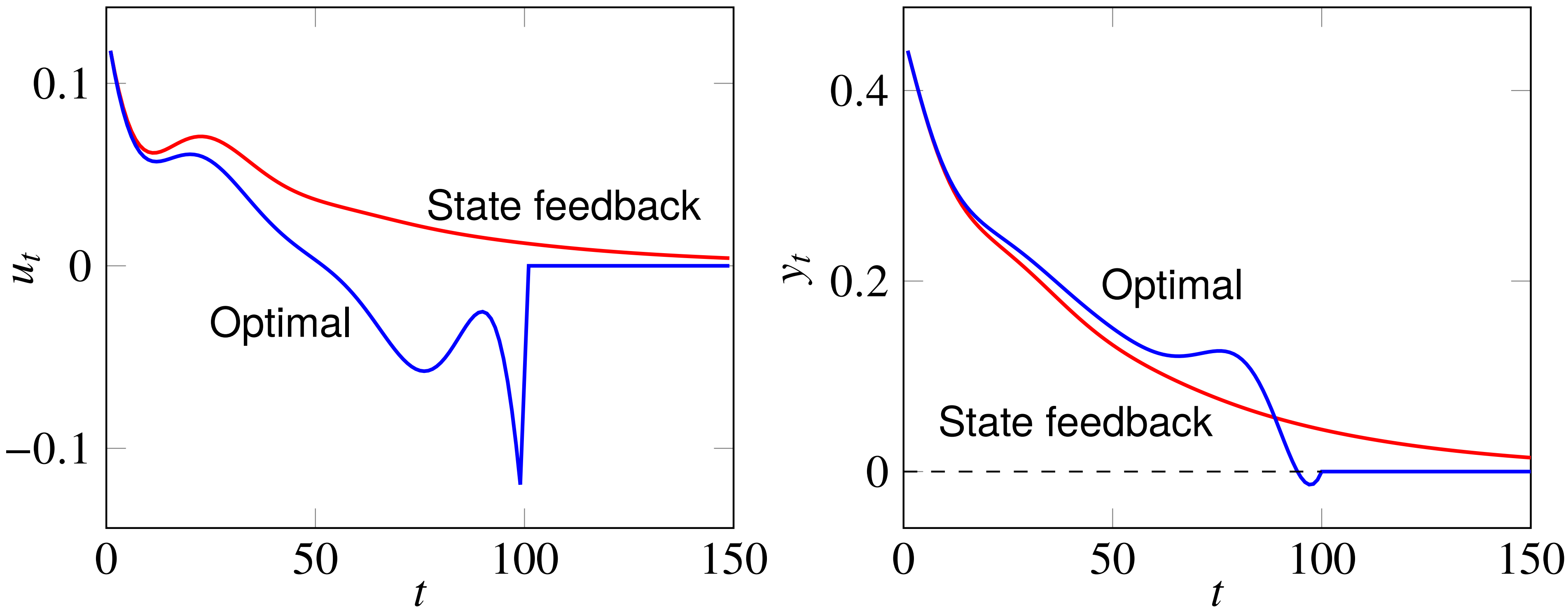
- K is *state feedback gain matrix*
- widely used, especially when x_t should converge to zero, T is not specified

- one choice for K: solve linear quadratic control problem with $x^{\text{des}} = 0$
- solution u_t is a linear function of x^{init}, so u_1 can be written as

$$u_1 = Kx^{\text{init}}$$

- columns of K can be found by computing u_1 for $x^{\text{init}} = e_1, \ldots, e_n$
- use this K as state feedback gain matrix

Example



- system matrices of previous example
- blue curve uses optimal linear quadratic control for $T = 100$
- red curve uses simple linear state feedback $u_t = Kx_t$

Outline

Portfolio optimization

Linear quadratic control

Linear quadratic state estimation

State estimation

- linear dynamical system model:

$$x_{t+1} = A_t x_t + B_t w_t, \quad y_t = C_t x_t + v_t, \quad t = 1, 2, \ldots$$

- x_t is *state* (n-vector)
- y_t is *measurement* (p-vector)
- w_t is *input* or *process noise* (m-vector)
- v_t is *measurement noise* or *measurement residual* (p-vector)
- we know A_t, B_t, C_t, and measurements $y_1, \ldots, y_T$
- w_t, v_t are unknown, but assumed small
- *state estimation*: estimate/guess $x_1, \ldots, x_T$

Least squares state estimation

$$
\begin{array}{ll}
\text{minimize} & J_{\text{meas}} + \lambda J_{\text{proc}} \\
\text{subject to} & x_{t+1} = A_t x_t + B_t w_t, \quad t = 1, \ldots, T-1
\end{array}
$$

- variables: states $x_1, \ldots, x_T$ and input noise $w_1, \ldots, w_{T-1}$
- primary objective J_{meas} is sum of squares of measurement residuals:

$$
J_{\text{meas}} = \|C_1 x_1 - y_1\|^2 + \cdots + \|C_T x_T - y_T\|^2
$$

- secondary objective J_{proc} is sum of squares of process noise

$$
J_{\text{proc}} = \|w_1\|^2 + \cdots + \|w_{T-1}\|^2
$$

- $\lambda > 0$ is a parameter, trades off measurement and process errors

Constrained least squares formulation

$$\begin{array}{ll} \text{minimize} & \|C_1x_1 - y_1\|^2 + \cdots + \|C_Tx_T - y_T\|^2 + \lambda(\|w_1\|^2 + \cdots + \|w_{T-1}\|^2) \\ \text{subject to} & x_{t+1} = A_tx_t + B_tw_t, \quad t = 1, \ldots, T-1 \end{array}$$

- can be written as

$$\begin{array}{ll} \text{minimize} & \|\tilde{A}z - \tilde{b}\|^2 \\ \text{subject to} & \tilde{C}z = \tilde{d} \end{array}$$

- vector z contains the $Tn + (T-1)m$ variables:

$$z = (x_1, \ldots, x_T, w_1, \ldots, w_{T-1})$$

Constrained least squares formulation

$$\tilde{A} = \left[\begin{array}{cccc|ccc} C_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_T & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \sqrt{\lambda} I & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \sqrt{\lambda} I \end{array}\right], \qquad \tilde{b} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_T \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cccccc|cccc} A_1 & -I & 0 & \cdots & 0 & 0 & B_1 & 0 & \cdots & 0 \\ 0 & A_2 & -I & \cdots & 0 & 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{T-1} & -I & 0 & 0 & \cdots & B_{T-1} \end{array}\right], \qquad \tilde{d} = 0$$

Missing measurements

- suppose we have measurements y_t for $t \in \mathcal{T}$, a subset of $\{1, \ldots, T\}$
- measurements for $t \notin \mathcal{T}$ are missing
- to estimate states, use same formulation but with

$$J_{\text{meas}} = \sum_{t \in \mathcal{T}} \|C_t x_t - y_t\|^2$$

- from estimated states $\hat{x}_t$, can estimate missing measurements

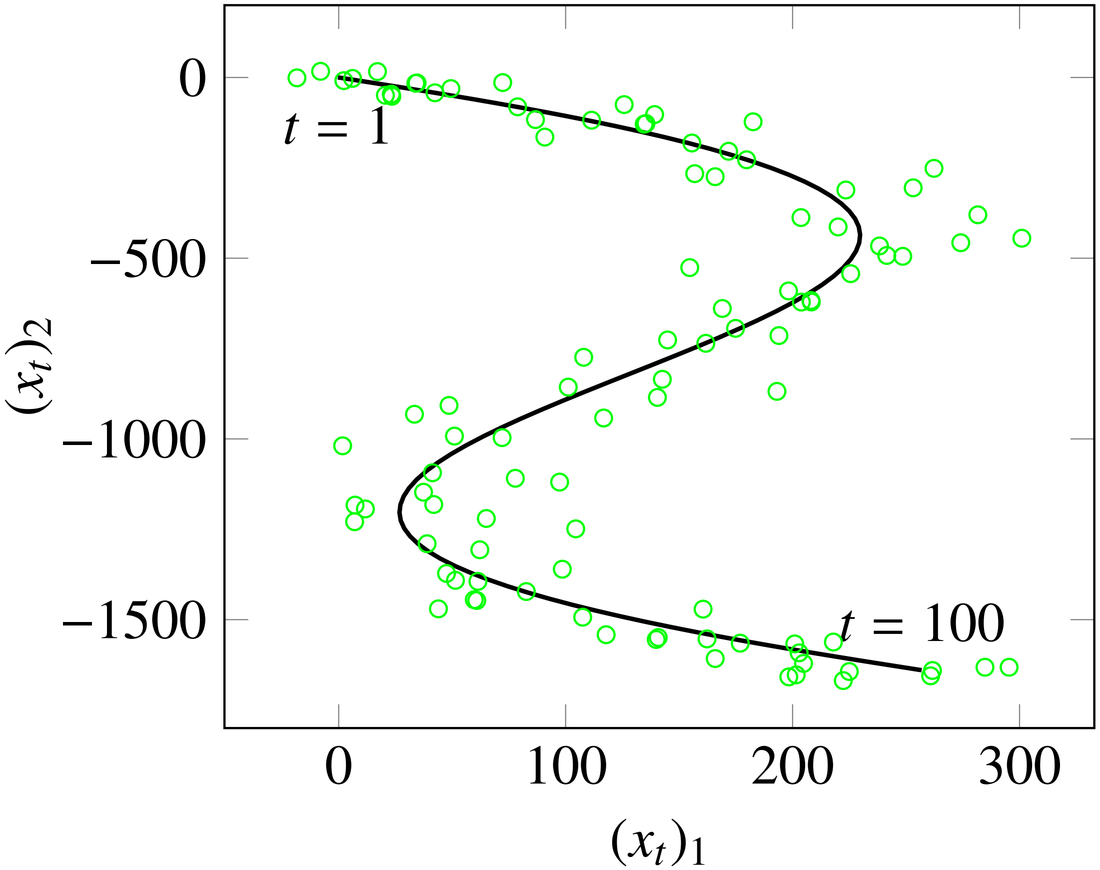
$$\hat{y}_t = C_t \hat{x}_t, \quad t \notin \mathcal{T}$$

Example

$$A_t = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B_t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

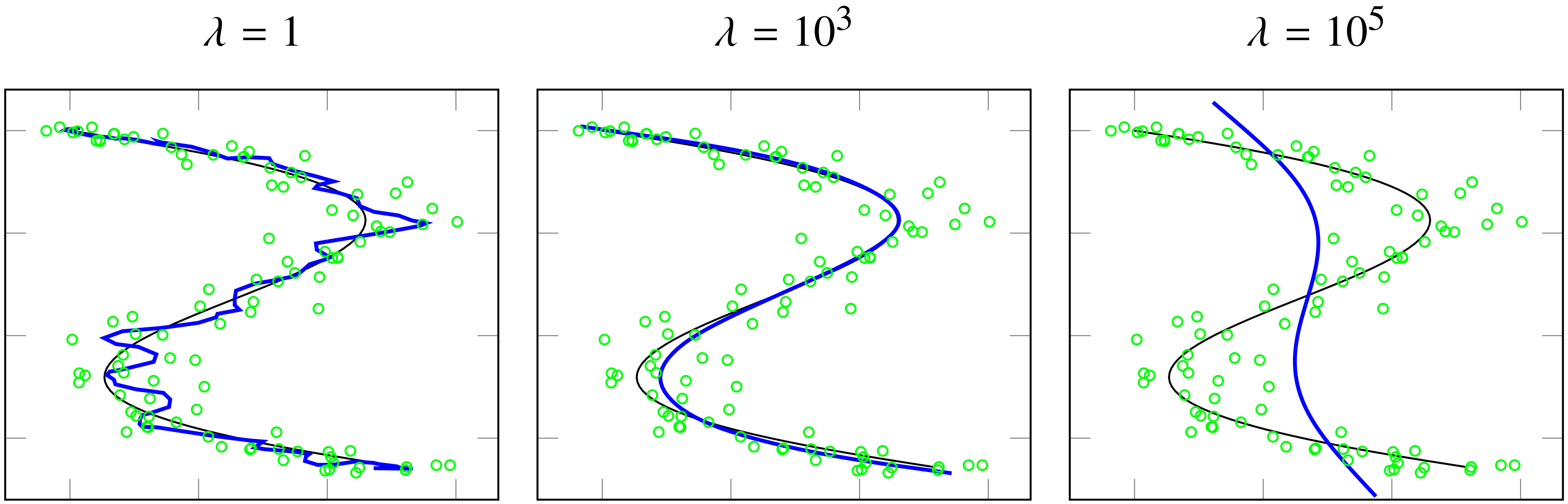
- simple model of mass moving in a 2-D plane
- $x_t = (p_t, z_t)$: 2-vector p_t is position, 2-vector z_t is the velocity
- $y_t = C_t x_t + w_t$ is noisy measurement of position
- $T = 100$

Measurements and true positions



- solid line is exact position $C_t x_t$
- 100 noisy measurements y_t shown as circles

Position estimates

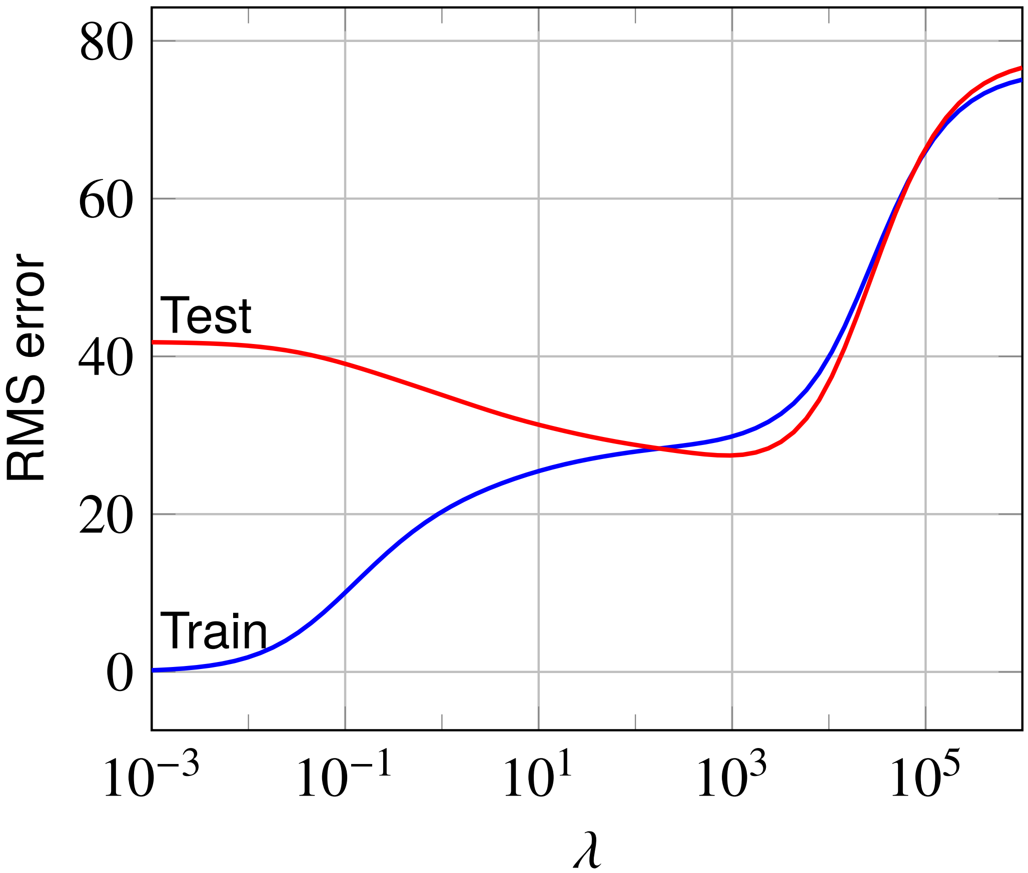


blue lines show position estimates for three values of λ

Cross-validation

- randomly remove 20% (say) of the measurements and use as test set
- for many values of λ, estimate states using other (*training*) measurements
- for each λ, evaluate RMS measurement residuals on test set
- choose λ to (approximately) minimize the RMS test residuals

Example



- cross-validation method applied to previous example
- remove 20 of the 100 measurements
- suggests using $\lambda \approx 10^3$