7. Matrix examples

Outline

Geometric transformations

Selectors

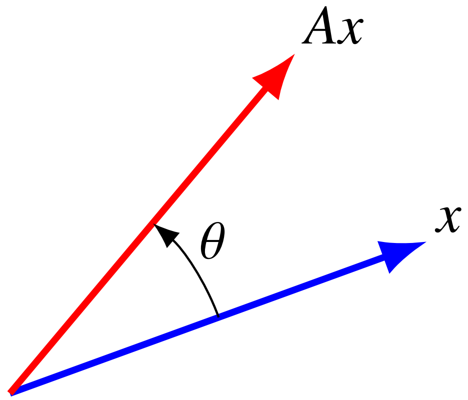
Incidence matrix

Convolution

Geometric transformations

- many geometric transformations and mappings of 2-D and 3-D vectors can be represented via matrix multiplication $y = Ax$
- for example, rotation by θ:

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$



(to get the entries, look at Ae_1 and Ae_2)

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Selectors

- an $m \times n$ *selector matrix*: each row is a unit vector (transposed)

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix}$$

- multiplying by A selects entries of x:

$$Ax = (x_{k_1}, x_{k_2}, \ldots, x_{k_m})$$

- example: the $m \times 2m$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

  'down-samples' by 2: if x is a $2m$-vector then $y = Ax = (x_1, x_3, \ldots, x_{2m-1})$

- other examples: image cropping, permutation, ...

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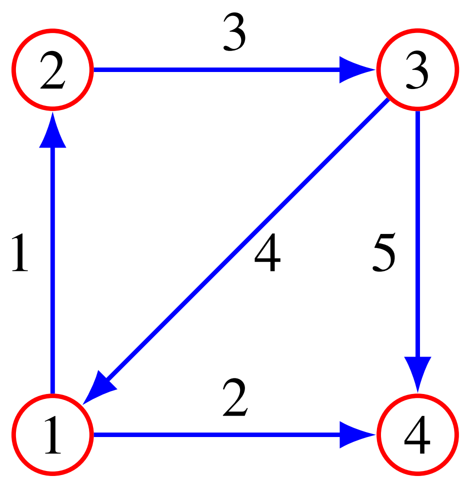
Incidence matrix

- graph with n vertices or nodes, m (directed) edges or links
- incidence matrix is $n \times m$ matrix

$$A_{ij} = \begin{cases} 1 & \text{edge } j \text{ points to node } i \\ -1 & \text{edge } j \text{ points from node } i \\ 0 & \text{otherwise} \end{cases}$$

- example with $n = 4$, $m = 5$:



$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Flow conservation

- m-vector x gives flows (of something) along the edges
- examples: heat, money, power, mass, people, ...
- $x_j > 0$ means flow follows edge direction
- Ax is n-vector that gives the total or net flows
- $(Ax)_i$ is the net flow into node i
- $Ax = 0$ is *flow conservation*; x is called a *circulation*

Potentials and Dirichlet energy

- suppose v is an n-vector, called a *potential*
- v_i is potential value at node i
- $u = A^T v$ is an m-vector of *potential differences* across the m edges
- $u_j = v_l - v_k$, where edge j goes from k to node l
- *Dirichlet energy* is $\mathcal{D}(v) = \|A^T v\|^2$,

$$\mathcal{D}(v) = \sum_{\text{edges } (k,l)} (v_l - v_k)^2$$

  (sum of squares of potential differences across the edges)
- $\mathcal{D}(v)$ is small when potential values of neighboring nodes are similar

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Convolution

- for n-vector a, m-vector b, the *convolution* $c = a * b$ is the $(n+m-1)$-vector

$$c_k = \sum_{i+j=k+1} a_i b_j, \quad k = 1, \ldots, n+m-1$$

- for example with $n = 4$, $m = 3$, we have

$$\begin{aligned}
c_1 &= a_1 b_1 \\
c_2 &= a_1 b_2 + a_2 b_1 \\
c_3 &= a_1 b_3 + a_2 b_2 + a_3 b_1 \\
c_4 &= a_2 b_3 + a_3 b_2 + a_4 b_1 \\
c_5 &= a_3 b_3 + a_4 b_2 \\
c_6 &= a_4 b_3
\end{aligned}$$

- example: $(1, 0, -1) * (2, 1, -1) = (2, 1, -3, -1, 1)$

Polynomial multiplication

- a and b are coefficients of two polynomials:

$$p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}, \qquad q(x) = b_1 + b_2 x + \cdots + b_m x^{m-1}$$

- convolution $c = a * b$ gives the coefficients of the product $p(x)q(x)$:

$$p(x)q(x) = c_1 + c_2 x + \cdots + c_{n+m-1} x^{n+m-2}$$

- this gives simple proofs of many properties of convolution; for example,

$$a * b = b * a$$

$$(a * b) * c = a * (b * c)$$

$$a * b = 0 \text{ only if } a = 0 \text{ or } b = 0$$

Toeplitz matrices

- can express $c = a * b$ using matrix-vector multiplication as $c = T(b)a$, with

$$T(b) = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ b_3 & b_2 & b_1 & 0 \\ 0 & b_3 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix}$$

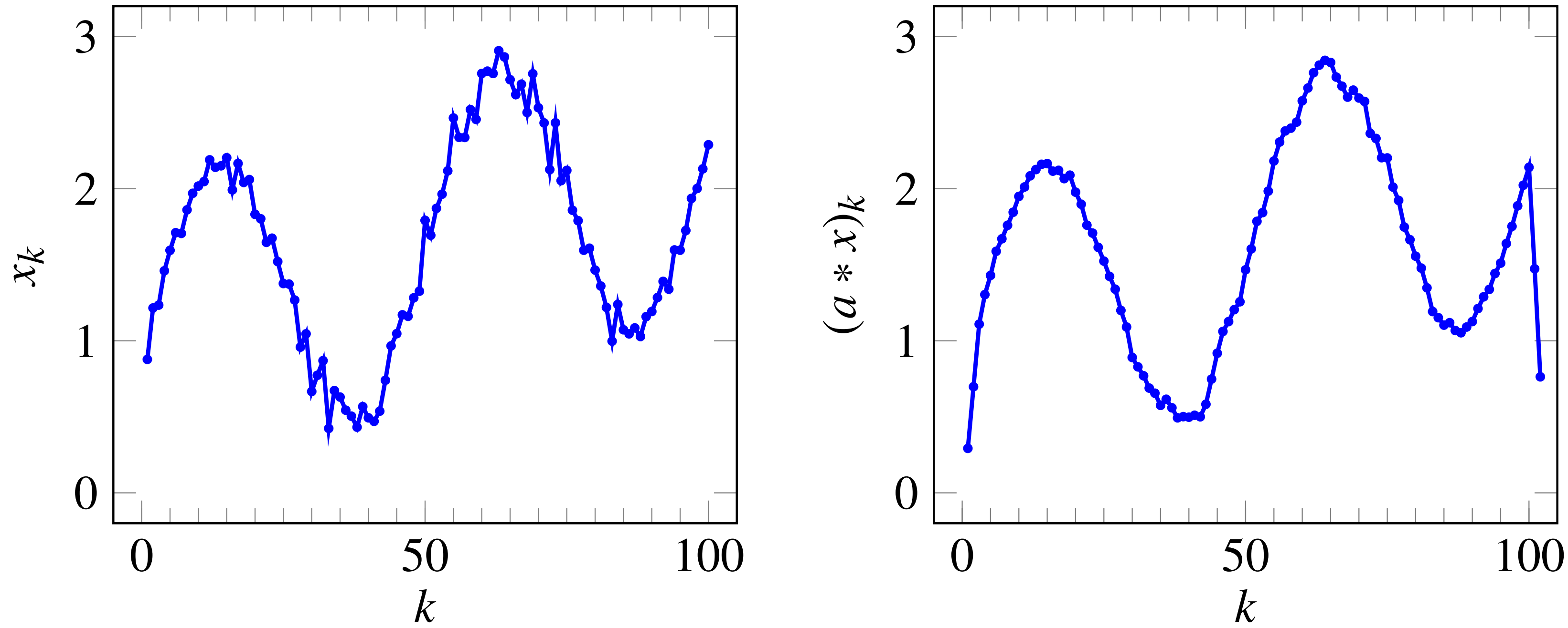
- $T(b)$ is a Toeplitz matrix (values on diagonals are equal)

Moving average of time series

- n-vector x represents a time series
- convolution $y = a * x$ with $a = (1/3, 1/3, 1/3)$ is 3-period *moving average*:

$$y_k = \frac{1}{3}(x_k + x_{k-1} + x_{k-2}), \quad k = 1, 2, \ldots, n+2$$

(with x_k interpreted as zero for $k < 1$ and $k > n$)



Input-output convolution system

- m-vector u represents a time series *input*
- $m+n-1$ vector y represents a time series *output*
- $y = h * u$ is a *convolution model*
- n-vector h is called the *system impulse response*
- we have

$$y_i = \sum_{j=1}^{n} u_{i-j+1} h_j$$

  (interpreting u_k as zero for $k < n$ or $k > n$)
- interpretation: y_i, output at time i is a linear combination of $u_i, \ldots, u_{i-n+1}$
- h_3 is the factor by which current output depends on what the input was 2 time steps before