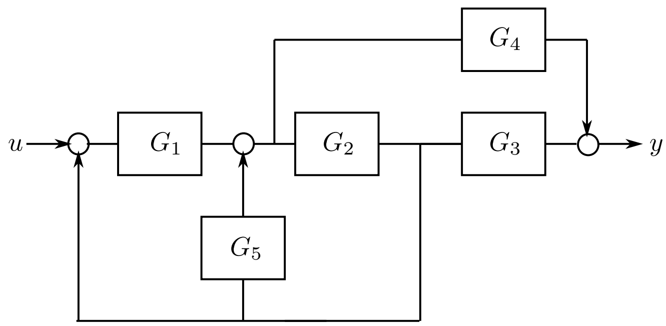
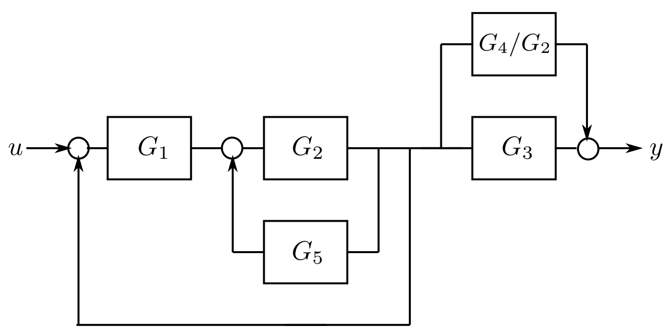
EE363 Automatic Control: Midterm Exam (4 problems, 90 minutes)

1) *Block diagram simplification (10 points).* Simplify the following block diagram. ,*i.e.*, find the transfer function from u to y. Your answer should be in terms of $G_1, \ldots, G_5$. You may assume $G_1, \ldots, G_5$ are scalar transfer functions.



Solution: The block diagram rearranges to



Hence we have

$$\frac{y(s)}{u(s)} = (G_3 + G_4/G_2)\,\frac{G_2G_1/(1 - G_2G_5)}{1 - G_2G_1/(1 - G_2G_5)}$$
$$= \frac{G_3G_2G_1 + G_4G_1}{1 - G_2G_5 - G_2G_1}$$

2) *Undershoot (10 points).* The step response is said to have an undershoot if it initially starts off in the *wrong* direction. We consider the following stable linear dynamical system with $n = m+1 > 10$, and simply assume that all the poles and zeros are real.

$$G(s) = -\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Suppose that there are an *odd* number of real right-half-plane zeros, for example, say, five (or other odd number of) zeros are on the positive real axis and the rest are on the other side. Show that the step response of $G(s)$ has an undershoot.

Solution: We have $p_1, p_2, \ldots, p_n > 0$ since $G(s)$ is stable. Also, there are an odd number of negative z_i's for $i = 1, \ldots, m$. Hence we have $p_1 p_2 \cdots p_n > 0$ and $z_1 z_2 \cdots z_m < 0$.

Applying the Final Value Theorem to the step response of $G(s)$ gives

$$\lim_{s\to 0} sG(s)\frac{1}{s} = -\frac{z_1 z_2 \cdots z_m}{p_1 p_2 \cdots p_n} > 0$$

and applying the Initial Value Theorem to the time derivative of the step response of $G(s)$ gives

$$\lim_{s\to\infty} ssG(s)\frac{1}{s} = -1$$

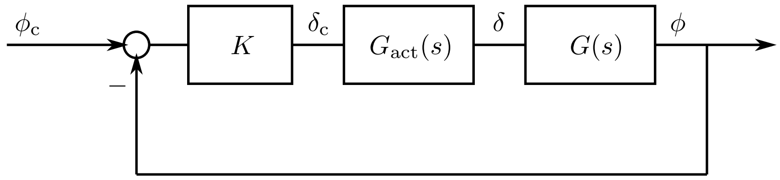
Therefore the step response of $G(s)$ starts off to the negative direction and converges to a positive value; it has an undershoot.

3) *Conservative control policy (10 points).* We consider a simple roll control system for a sounding rocket. A simple roll dynamics, which describes *the rotation of the rocket around its axisymmetric axis*, is shown below

$$\dot{p} = L_p p + L_\delta \delta$$
$$\dot{\phi} = p$$

where the state variables, ϕ and p, are the roll angle and the roll rate, respectively, and the input to the system, δ is the fin deflection angle. The coeffcient $L_p < 0$ is called *the aerodynamic damping* which originates from the aerodynamic friction that tends to stop the rotation, and L_δ can be interpreted as *the control effectiveness* of the control fin.

The control effectiveness L_δ is given accurately by $L_\delta = 1$, however the problem we have is that the aerodynamic damping term, L_p is relatively inaccurate; it can be somewhere between -2 and -1, *i.e.*, $-2 \leq L_p \leq -1$.



The above block diagram depicts the control system that we will work on. Your constant gain controller, K, computes the control command δ_c by using the roll tracking error which is the difference between the reference roll command ϕ_c and the actual roll angle ϕ. The computed control command δ_c is sent to the actuator $G_{act}(s)$ which finally generates the control fin angle δ. We assume that the actuator is ideal, *i.e.*, $G_{act}(s) = 1$.

Design a constant gain controller K such that its closed loop response is as fast as possible, while guaranteeing the closed loop damping to be greater than or equal to $1/\sqrt{2}$ for all $G(s)$ under $-2 \leq L_p \leq -1$.

Solution: The closed loop transfer function from ϕ_c to ϕ is

$$\frac{\phi(s)}{\phi_c(s)} = \frac{K}{s^2 - L_p s + K}$$

which gives the closed loop damping as

$$\zeta = -\frac{L_p}{2\sqrt{K}}$$

For $-2 \leq L_p \leq -1$, the closed loop damping lies between

$$\frac{1}{2\sqrt{K}} \leq \zeta \leq \frac{1}{\sqrt{K}}$$

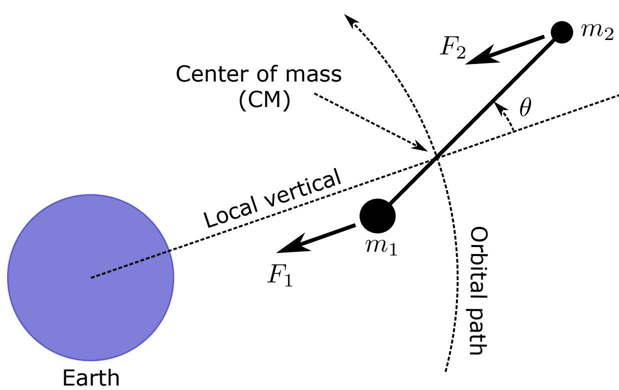
so the largest K that guarantees $\zeta > 1/\sqrt{2}$ happens when

$$\frac{1}{2\sqrt{K}} = \frac{1}{\sqrt{2}}$$

which is simply when $K = 1/2$.

4) *Gravity gradient stabilization (10 points).* The picture below shows *Uribyol-1*, the first Korean satellite which was launched on August 11th, 1992. An interesting feature that we are going to look at in this problem is its long expandable boom on top, which is used as a passive attitude control system that lets the satellite librate (oscillate) around the local vertical axis so that its main body always faces the Earth.

We model the planar motion of the orbiting Uribyol by the following two bodies of mass m_1 and m_2 connected by a stiff massless rod with length l. Since the gravitational force acting on a body is inversely proportional to its squared distance to the Earth's center, the gravitational forces acting on m_1 and m_2 are slightly different, and this difference generates a tiny amount of restoring torque. Although the level of the torque is quite low, it is adequate to coarsely stabilize Uribyol in space where the other external disturbances are way smaller.



a) Find the equation of motion describing the dynamics of the libration angle, θ. You may simply assume that the center of mass (CM) is fixed in space (ignore the revolution). The following information will be helpful:
   - The libration angle θ is small.
   - Gravitational force acting on a body of mass m at distance d: $F = \mu m/d^2$
   - Distance from m_1 to the Earth's center: r
   - Length from m_1 to Uribyol's CM: $m_2 l/(m_1 + m_2)$
   - Uribyol's moment of inertia about its CM: $J = m_1 m_2 l^2/(m_1 + m_2)$

b) What is the libration frequency of θ? Your answer should be in terms of μ, r, and l.

c) Further reduce the answer you obtained in (b), by using the approximation $r \gg l$. Hint. You will see something like the *gravity gradient* from your answer.

Solution:

a) The equation of motion describing the dynamics of θ under small angle approximation:

$$\begin{aligned} J\ddot{\theta} = \frac{m_1 m_2 l^2}{m_1 + m_2}\ddot{\theta} &= \frac{\mu m_2}{(r+l)^2}\frac{m_1 l}{m_1 + m_2}\theta - \frac{\mu m_1}{r^2}\frac{m_2 l}{m_1 + m_2}\theta \\ &= \frac{\mu m_1 m_2 l}{m_1 + m_2}\left(\frac{1}{(r+l)^2} - \frac{1}{r^2}\right)\theta \end{aligned}$$

which simplifies to

$$\ddot{\theta} + \frac{\mu(2r+l)}{r^2(r+l)^2}\theta = 0$$

b) The libration frequency ω is

$$\omega = \sqrt{\frac{\mu(2r+l)}{r^2(r+l)^2}} = \sqrt{\frac{\mu(2+l/r)}{r^3(1+l/r)^2}}$$

c) Approximation of $r \gg l$ leads to

$$\omega = \sqrt{\frac{2\mu}{r^3}}$$

For your information, the gravitational acceleration acting on a body of mass m at distance r is equal to $g = \mu/r^2$ (since the force is equal to $F = \mu m/r^2$). Thus the gravity gradient is given by

$$\frac{\partial g}{\partial r} = \frac{\partial}{\partial r}\frac{\mu}{r^2} = -\frac{2\mu}{r^3}$$

which is why the technique is called the *gravity gradient stabilization*.

5) *Midterm statistics.*