ASE6029 Linear optimal control: Homework #3

1) *LQR with affine dynamics.* Suppose $Q_0, \ldots, Q_N \geq 0$, $R_0, \ldots, R_{N-1} > 0$, and consider the following linear quadratic regulator design problem under affine dynamical constraints with A, B, and b.

$$\begin{array}{ll} \underset{u_0,\ldots,u_{N-1}}{\text{minimize}} & \displaystyle\sum_{k=0}^{N-1} \left(x_k^T Q_k x_k + u_k^T R_k u_k\right) + x_N^T Q_N x_N \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + b, \quad \forall k \in \{0, \ldots, N-1\} \end{array}$$

Show that the optimal solution is affine in x and is explicitly given by

$$u_k = K_k x_k + l_k$$

where the control gains are given by

$$\begin{aligned} K_k &= -\left(B^T P_{k+1} B + R_k\right)^{-1} B^T P_{k+1} A \\ l_k &= -\left(B^T P_{k+1} B + R_k\right)^{-1} B^T \left(P_{k+1} b + q_{k+1}\right) \end{aligned}$$

with

$$\begin{aligned} P_k &= Q_k + A^T P_{k+1} A - A^T P_{k+1} B \left(B^T P_{k+1} B + R_k\right)^{-1} B^T P_{k+1} A \\ q_k &= (A + BK_k)^T \left(P_{k+1} b + q_{k+1}\right) \end{aligned}$$

computed by backward recursion from $P_N = Q_N$ and $q_N = 0$.

Hint: Assume that the value function at step k is quadratic with

$$\begin{aligned} V_k(z) &= z^T P_k z + 2q_k^T z + r_k \\ &= \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} P_k & q_k \\ q_k^T & r_k \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}. \end{aligned}$$