ASE6029 Linear optimal control: Homework #5

1) *Feedback invariants.* Given a continuous-time linear time invariant (LTI) system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

we say that a functional

$$H(x(\cdot), u(\cdot))$$

that involves the system's input and state is a *feedback invariant* for the system if, when computed along a solution to the system (1), its value depends only on the initial condition $x(0)$ and not on the specific input signal $u(t)$.

Show that, for every symmetric matrix P, the functional

$$H(x(\cdot), u(\cdot)) = -\int_0^\infty (Ax(t) + Bu(t))^T Px(t) + x(t)^T P(Ax(t) + Bu(t))dt \tag{2}$$

is a feedback invariant for system (1) as long as $\lim_{t\to\infty} x(t) = 0$.

2) *Feedback invariants in optimal control.* Suppose that we are able to express a cost function $J(x(\cdot), u(\cdot))$ to be minimized by an appropriate choice of the input $u(t)$ in the following form:

$$J(x(\cdot), u(\cdot)) = H(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda(x(t), u(t))dt. \tag{3}$$

where $H(x(\cdot), u(\cdot))$ is a feedback invariant and the function $\Lambda(x(t), u(t))$ has the property that

$$\min_{u(t)} \Lambda(x(t), u(t)) = 0, \qquad \text{for all } x(t).$$

In this case, the control

$$u^*(t) = \arg\min_{u(t)} \Lambda(x(t), u(t)),$$

minimizes the functional (3), and the optimal value of (3) is equal to the feedback invariant

$$J(x(\cdot), u(\cdot)) = H(x(\cdot), u(\cdot)).$$

The LQR cost can be expressed as in (3) with the feedback invariant in (2), provided that we choose the matrix P appropriately. To check that this is so, we add and subtract this feedback invariant to the LQR cost and conclude that

$$\begin{aligned} J_{\text{lqr}}(x(\cdot), u(\cdot)) &= \int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)\ dt \\ &= H_{\text{lqr}}(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda_{\text{lqr}}(x(t), u(t))dt. \end{aligned}$$

a) What is $\Lambda_{\text{lqr}}(x(t), u(t))$? Your answer should be in terms of A, B, Q, R, and P (as well as $x(t)$ and $u(t)$).

b) Separate $\Lambda_{\text{lqr}}(x(t), u(t))$ into two parts,

$$\Lambda_{\text{lqr}}(x(t), u(t)) = \Lambda_0(x(t)) + \Lambda_+(x(t), u(t))$$

where $\Lambda_0(x(t))$ depends only on $x(t)$ and not on $u(t)$, and $\Lambda_+(x(t), u(t))$ is nonnegative for all $x(t)$ and $u(t)$.

c) Explain how you can choose $u^*(t)$ and P such that $\Lambda_{\text{lqr}}(x(t), u(t)) = 0$ for all $x(t)$.

d) What is the optimal LQR cost in this case? What happens if P is not positive semidefinite?

3) *Controllability, observability, and Hamiltonian.* Consider the following continuous-time LTI system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

a) Show that when the matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is full column rank, then the matrix

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

is full column rank for every $\lambda \in \mathbb{C}$. *Hint: Prove the statement by contradiction.* Your answer should also prove that, when the matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is full row rank, then the matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

is full row rank for every $\lambda \in \mathbb{C}$.

b) Show that if $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$ with $v_1, v_2 \in \mathbb{C}^n$ is an eigenvector of a matrix $H \in \mathbb{R}^{2n \times 2n}$ associated with an eigenvalue $\lambda = j\omega$ on the imaginary axis, then

$$\begin{bmatrix} v_2^* & v_1^* \end{bmatrix} Hv + (Hv)^* \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = 0,$$

where $(\cdot)^*$ denotes the complex conjugate transpose. Notice that the order of the indexes of v_1 and v_2 above is opposite to the order in the definition of v.

c) Show that if $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$ with $v_1, v_2 \in \mathbb{C}^n$ is an eigenvector of

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

associated with an eigenvalue $\lambda = j\omega$ on the imaginary axis, then

$$B^T v_2 = 0 \quad \text{and} \quad C v_1 = 0.$$

d) Show that, if for every $\lambda \in \mathbb{C}$

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \quad \text{is full row rank}$$

and

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \quad \text{is full column rank}$$

then the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

cannot have any eigenvalues on the imaginary axis.

The results we've got here imply that, if (A, B) is controllable and (C, A) is observable, the Hamiltonian matrix associated with the system has exactly n stable eigenvalues and n unstable eigenvalues. Together with what we've discussed in class, this in turn implies that there exists a unique stabilizing P that makes $A + BK = A - BR^{-1}B^T P$ a stability matrix (all of whose eigenvalues have strictly negative real parts).