ASE7030 Convex optimization: Homework #2

1) *Least squares solution using SVD.* Consider a tall rank deficient matrix $A \in \mathbb{R}^{m\times n}$ with $m > n$ is given and its singular value decomposition is given by:

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T$$

where $\operatorname{rank} A = r < n$, $U_1 \in \mathbb{R}^{m\times r}$, $\Sigma_1 = R^{r\times r}$, and $V_1 \in \mathbb{R}^{n\times r}$.

Show that all solutions for:

$$\text{minimize} \quad \|Ax - b\|^2$$

is parameterized as follows with some $z \in \mathbb{R}^{n-r}$:

$$x^{\text{lsqr}} = V_1 \Sigma_1^{-1} U_1^T b + V_2 z$$

2) *Least norm solution using SVD.* Consider a wide rank deficient matrix $A \in \mathbb{R}^{m\times n}$ with $m < n$ is given and its singular value decomposition is given by:

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T$$

where $\operatorname{rank} A = r < m$, $U_1 \in \mathbb{R}^{m\times r}$, $\Sigma_1 = R^{r\times r}$, and $V_1 \in \mathbb{R}^{n\times r}$. Suppose that $b \in \operatorname{range} A$ so there exists at least one x that satisfy $Ax = b$; in fact it has infinitely many solutions.

a) Show that all solutions for $Ax = b$ is parameterized with $z \in \mathbb{R}^{n-r}$ by:

$$x^* = V_1 \Sigma_1^{-1} U_1^T b + V_2 z$$

b) Show that, among all solutions that satisfy $Ax = b$, the one with smallest $\|x\|^2$ is uniquely given by:

$$x^{\text{ln}} = V_1 \Sigma_1^{-1} U_1^T b$$

In other words, the optimal solution for the following:

$$\begin{array}{ll} \text{minimize} & \|x\|^2 \\ \text{subject to} & Ax = b \end{array}$$

is unique, and is equal to x^{ln}.