Solving optimal control under nonconvex constraints via first order methods

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Optimization

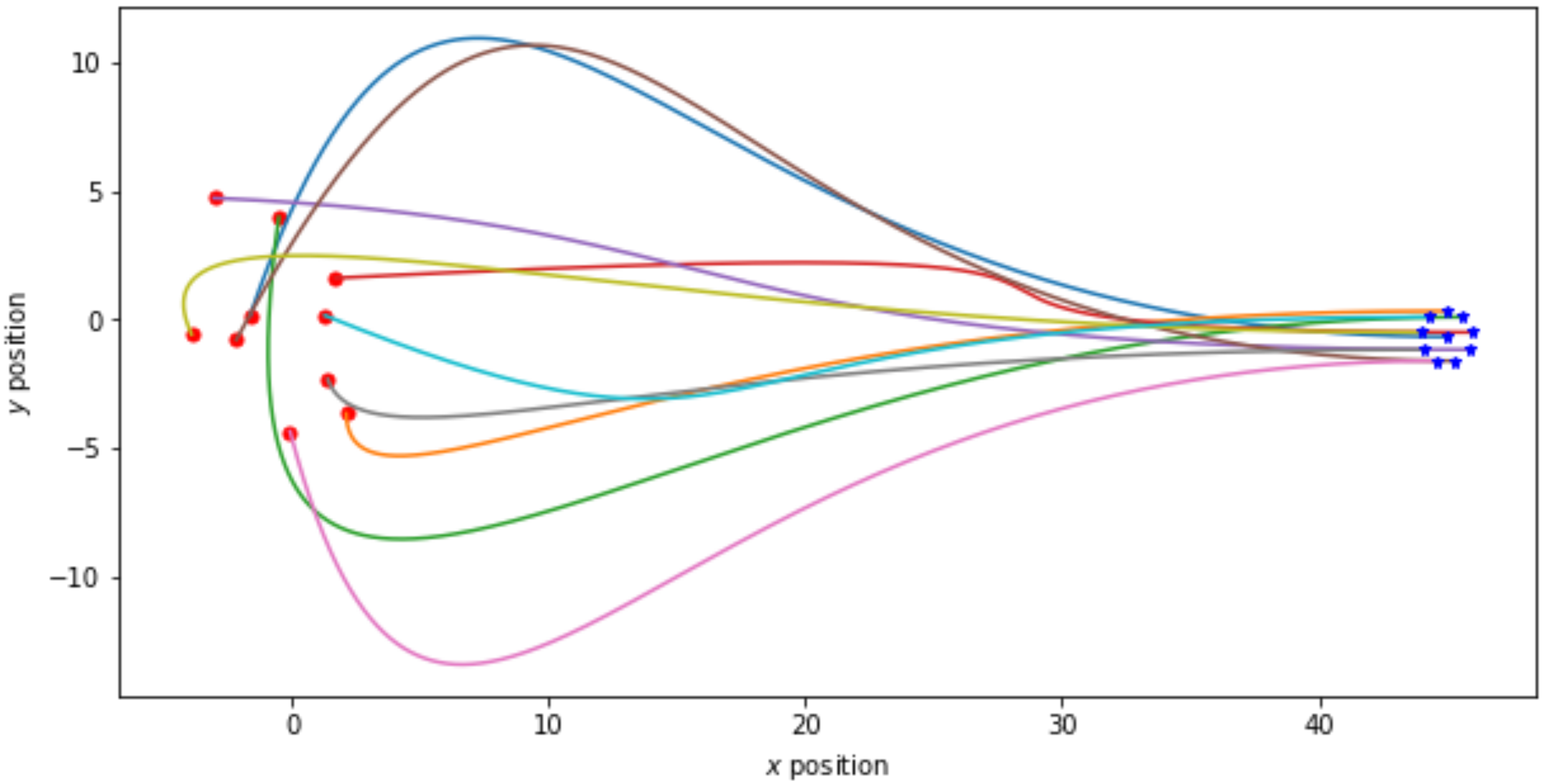
General optimization problems:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- Finding x that minimizes $f(x)$ while x being inside $\mathcal{C}$.
- Called a convex problem if $f(x)$ is convex and $\mathcal{C}$ is convex.
- We are interested in solving such problems with nonconvex $\mathcal{C}$.

Some examples in control

Multiple drones in formation:



- Finding minimum energy trajectories for formation shaping

Some examples in control

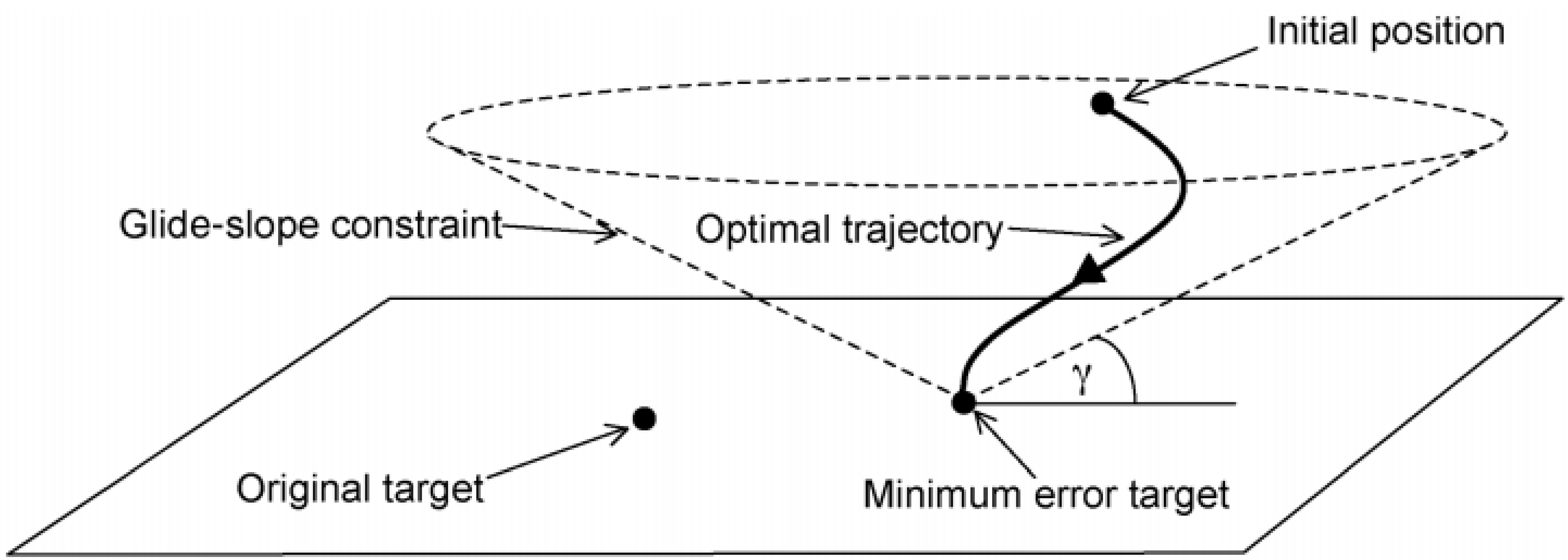
Multiple drones in formation:

$$\begin{array}{ll} \underset{u_t^{(k)}}{\text{minimize}} & \displaystyle\sum_{k=1}^{K}\sum_{t=0}^{T-1} \|u_t^{(k)}\|^2 \\ \text{subject to} & v_{t+1}^{(k)} = v_t^{(k)} - \gamma h v_t^{(k)} + h u_t^{(k)}, \\ & x_{t+1}^{(k)} = x_t^{(k)} + (1 - 0.5\gamma h) h v_t^{(k)} + 0.5 h^2 u_t^{(k)}, \\ & \|u_t^{(k)}\| \leq u_{\mathsf{ub}}, \qquad \|v_t^{(k)}\| \leq v_{\mathsf{ub}}, \\ & (v_T^{(1)}, \ldots, v_T^{(K)}) \in \mathcal{V}_{\mathsf{des}}, \\ & (x_T^{(1)}, \ldots, x_T^{(K)}) \in \mathcal{X}_{\mathsf{des}} \end{array}$$

- A convex problem

Some examples in control

Soft landing guidance:



- Optimal soft landing guidance for reusable launcher

Some examples in control

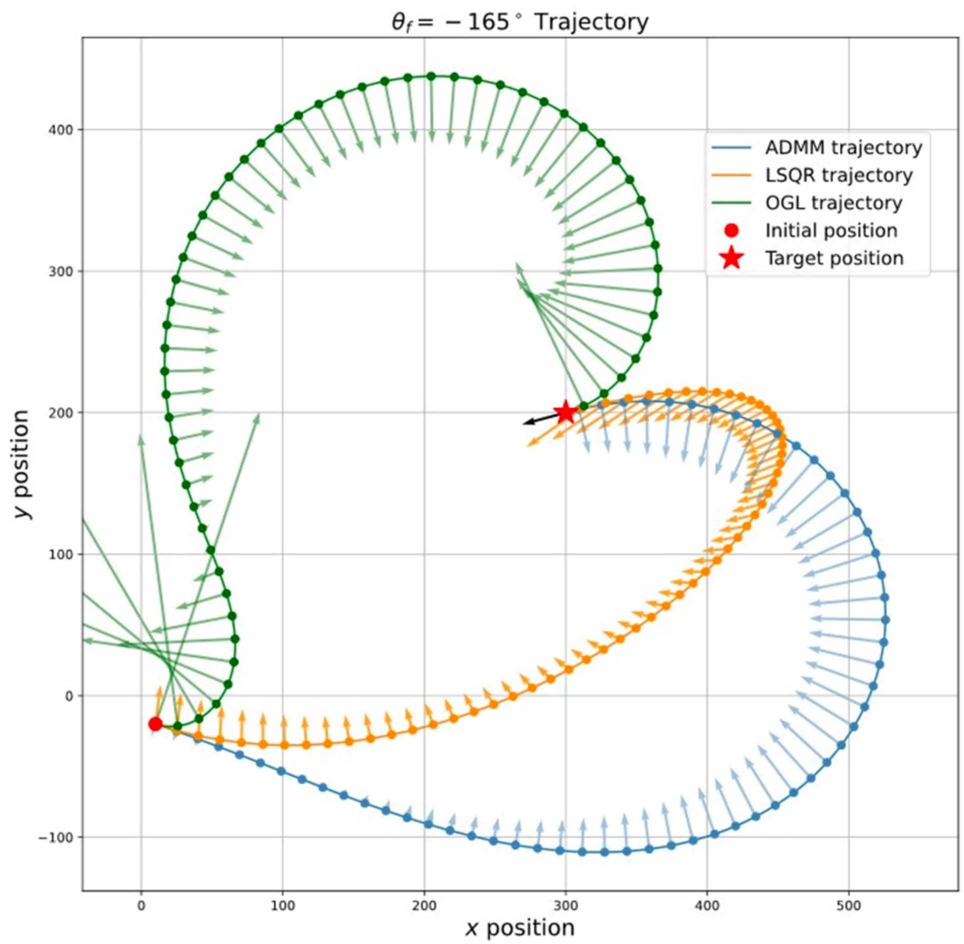
Soft landing guidance:

$$
\begin{array}{ll}
\underset{u(t)}{\text{minimize}} & \int_0^{t_f} \|u(t)\| dt \\
\text{subject to} & \ddot{r}(t) = g + u(t)/m(t), \\
& \dot{m}(t) = -\alpha \|u(t)\|, \\
& 0 < \rho_1 \leq \|u(t)\| \leq \rho_2, \\
& \|r_h(t)\| \leq \beta r_v(t), \\
& r(t_f) = r_{\text{des}}, \\
& \dot{r}(t_f) = 0
\end{array}
$$

- Nonconvex due to the lower bound in thrust

Some examples in control

Large divert missile guidance:



- Computing maneuver command for large divert impact angle guidance

Some examples in control

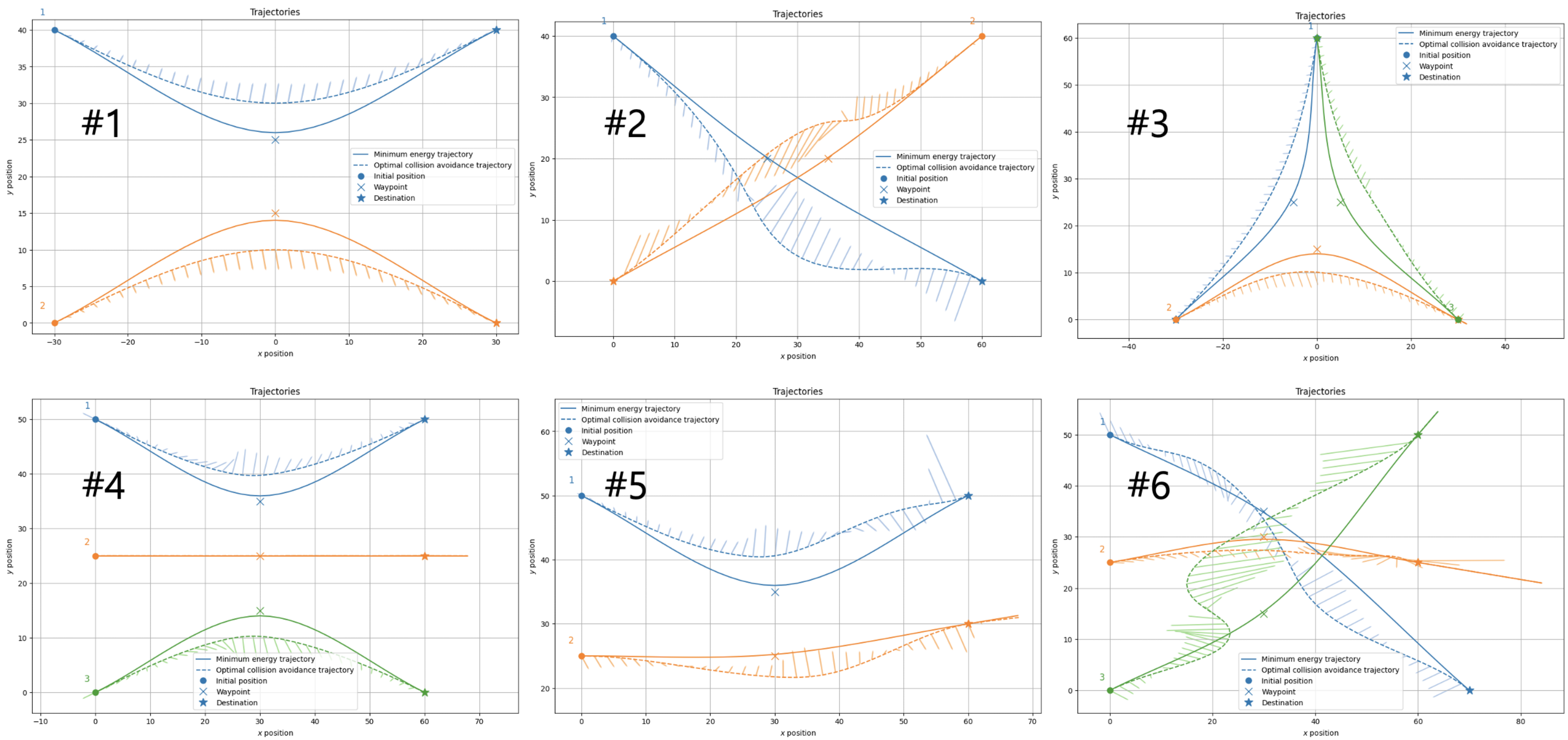
Large divert missile guidance:

$$
\begin{array}{ll}
\underset{u_t}{\text{minimize}} & \sum_{t=0}^{T-1} \|u_t\|^2 \\
\text{subject to} & v_{t+1} = v_t - \gamma h v_t + h u_t, \\
& x_{t+1} = x_t + (1 - 0.5\gamma h) h v_t + 0.5 h^2 u_t, \\
& \|u_t\| \le u_{\text{ub}}, \qquad \|v_t\| \le v_{\text{ub}}, \\
& v_T \in \mathcal{V}_{\text{des}}, \\
& x_T \in \mathcal{X}_{\text{des}}, \\
& \langle v_t, u_t \rangle = 0
\end{array}
$$

- Nonconvex due to the control-orthogonality condition

Some examples in control

Multiple drones with collision avoidance:



- Finding minimum energy trajectories for multiple players with collision avoidance

Some examples in control

Multiple drones with collision avoidance:

$$\begin{array}{ll} \underset{u_t^{(k)}}{\text{minimize}} & \sum_{k=1}^{K} \sum_{t=0}^{T-1} \|u_t^{(k)}\|^2 \\ \text{subject to} & v_{t+1}^{(k)} = v_t^{(k)} - \gamma h v_t^{(k)} + h u_t^{(k)}, \\ & x_{t+1}^{(k)} = x_t^{(k)} + (1 - 0.5\gamma h) h v_t^{(k)} + 0.5 h^2 u_t^{(k)}, \\ & \|u_t^{(k)}\| \le u_{\text{ub}}, \qquad \|v_t^{(k)}\| \le v_{\text{ub}}, \\ & (v_T^{(1)}, \ldots, v_T^{(K)}) \in \mathcal{V}_{\text{des}}, \qquad (x_T^{(1)}, \ldots, x_T^{(K)}) \in \mathcal{X}_{\text{des}}, \\ & \|x_t^{(k)} - x_t^{(l)}\| \ge d_{\textit{safety}} \qquad \text{for } l \neq k \end{array}$$

- Nonconvex due to the collision avoidance constraints

First order methods

First order methods for convex optimization:

- Only requires the first order derivative information
- Compared to the second order methods
  - easy to implement and robust
  - computationally cheap and memory efficient
  - suitable for large-scale or distributed optimization
  - but less accurate
- Related topics
  - alternating direction method of multipliers
  - proximal algorithms
  - operator splitting methods
  - monotone operators

First order methods

Alternating direction method of multipliers:

Standard form:

$$\begin{array}{ll} \underset{x,z}{\text{minimize}} & f(x) + g(z) \\ \text{subject to} & Ax + Bz = c \end{array}$$

- Optimizing a separable composite function under linear equality constraints
- $f(x)$ and $g(z)$ convex

First order methods

Alternating direction method of multipliers:

Augmented Lagrangian:

$$\begin{aligned} L_\rho(x,z,y) &= f(x) + g(z) + y^T(Ax+Bz-c) + (\rho/2)\|Ax+Bz-c\|^2 \\ &= f(x) + g(z) + (\rho/2)\|Ax+Bz-c+u\|^2 + \text{const.} \end{aligned}$$

with $u = y/\rho$.

ADMM iteration:

$$\begin{aligned} x^{k+1} &= \operatorname*{argmin}_x \left(f(x) + (\rho/2)\|Ax+Bz^k-c+u^k\|^2 \right) \\ z^{k+1} &= \operatorname*{argmin}_z \left(g(z) + (\rho/2)\|Ax^{k+1}+Bz-c+u^k\|^2 \right) \\ u^{k+1} &= u^k + (Ax^{k+1}+Bz^{k+1}-c) \end{aligned}$$

First order methods

Standard convex optimization problems:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

which is equivalent to:

$$\underset{x}{\text{minimize}} \quad f(x) + I_{\mathcal{C}}(x) \qquad \text{where } I_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{C} \\ \infty, & \text{otherwise} \end{cases}$$

and:

$$\begin{array}{ll} \underset{x,z}{\text{minimize}} & f(x) + I_{\mathcal{C}}(z) \\ \text{subject to} & x - z = 0 \end{array}$$

First order methods

Solving standard convex optimization problems via ADMM:

ADMM iteration:

$$
\begin{aligned}
x^{k+1} &= \operatorname*{argmin}_x \left(f(x) + (\rho/2)\|x - z^k + u^k\|^2 \right) \\
z^{k+1} &= \operatorname*{argmin}_z \left(I_{\mathcal{C}}(z) + (\rho/2)\|x^{k+1} - z + u^k\|^2 \right) \\
u^{k+1} &= u^k + x^{k+1} - z^{k+1}
\end{aligned}
$$

- x-update is a convex optimization problem.
- u-update is summation and subtraction.
- z-update?

First order methods

Proximal operator:

The z-update:

$$z^{k+1} = \operatorname*{argmin}_z \left(I_{\mathcal{C}}(z) + (\rho/2)\|x^{k+1} - z + u^k\|^2 \right)$$
$$= \operatorname*{argmin}_{z \in \mathcal{C}} \|x^{k+1} - z + u^k\|^2 = \Pi_{\mathcal{C}}(x^{k+1} + u^k)$$

FYI:

$$z^{k+1} = \mathrm{prox}_{I_{\mathcal{C}}}(x^{k+1} + u^k)$$

where

$$\mathrm{prox}_{\alpha f}(y) = \operatorname*{argmin}_x \left(\alpha f(x) + \frac{1}{2}\|x - y\|^2 \right)$$

- In fact, the x- and the z-update are some form of proximal operators.

First order methods

Solving standard convex optimization problems via ADMM:

With the z-step simplified we have:

$$
\begin{aligned}
x^{k+1} &= \operatorname*{argmin}_x \left(f(x) + (\rho/2)\|x - z^k + u^k\|^2 \right) \\
z^{k+1} &= \Pi_{\mathcal{C}}(x^{k+1} + u^k) \\
u^{k+1} &= u^k + x^{k+1} - z^{k+1}
\end{aligned}
$$

where $\Pi_{\mathcal{C}}(x)$ is the orthogonal projection of x onto $\mathcal{C}$.

- In general, computing $\Pi_{\mathcal{C}}(x)$ is another convex optimization problem.
- However in many cases, $\Pi_{\mathcal{C}}(x)$ can be computed exactly, easily and explicitly.
- *Furthermore, the same arguments hold even when $\mathcal{C}$ is nonconvex.*

Examples

Multiple drones with collision avoidance:

Original problem:

$$
\begin{array}{ll}
\underset{u_t^{(k)}}{\text{minimize}} & \displaystyle\sum_{k=1}^{K}\sum_{t=0}^{T-1} \|u_t^{(k)}\|^2 \\
\text{subject to} & v_{t+1}^{(k)} = v_t^{(k)} - \gamma h v_t^{(k)} + h u_t^{(k)}, \\
& x_{t+1}^{(k)} = x_t^{(k)} + (1 - 0.5\gamma h) h v_t^{(k)} + 0.5 h^2 u_t^{(k)}, \\
& x_T = x_{\text{des}}, \\
& \|x_t^{(k)} - x_t^{(l)}\| \geq d_{\textit{safety}} \qquad \text{for } l \neq k
\end{array}
$$

Examples

Multiple drones with collision avoidance:

Standard form:

$$
\begin{array}{ll}
\underset{u}{\text{minimize}} & \|u\|^2 \\
\text{subject to} & Au = b, \\
& G_i u + h_i \in \mathcal{C}, \qquad \text{for } i = 1, 2, \ldots, L = K(K-1)/2
\end{array}
$$

Standard ADMM form:

$$
\begin{array}{ll}
\underset{u, z_1, \ldots, z_L}{\text{minimize}} & \|u\|^2 + I_{\mathcal{C}}(z_1) + \cdots + I_{\mathcal{C}}(z_L) \\
\text{subject to} & Au + b = 0, \\
& z_i = G_i u + h_i, \qquad \text{for } i = 1, 2, \ldots, L = K(K-1)/2
\end{array}
$$

Examples

Multiple drones with collision avoidance:

Solution approaches:

$$u^{k+1} = \operatorname*{argmin}_u \left\{ \|u\|^2 + \frac{\rho}{2}\left(\|Au + b + w_0^k\|^2 + \sum_{i=1}^{L} \|G_i u - z_i^k + h_i + w_i^k\|^2 \right) \right\}$$

$$z_i^{k+1} = \Pi_{\mathcal{C}}(G_i u^{k+1} + h_i + w_i^k), \qquad \text{for } i = 1, \ldots, L$$

$$w_0^{k+1} = w_0^k + Au^{k+1} + b$$

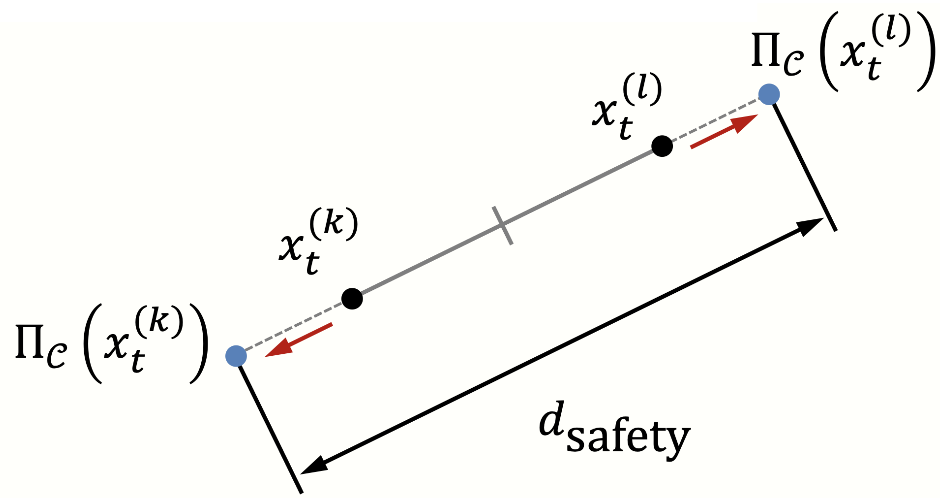
$$w_i^{k+1} = w_i^k + G_i u^{k+1} - z_i^{k+1} + h_i, \qquad \text{for } i = 1, \ldots, L$$

Examples

Problems with collision avoidance constraints:

$$\|x_t^{(k)} - x_t^{(l)}\| \geq d_{\mathsf{safety}}$$

Projection onto collision avoidance constraint set:

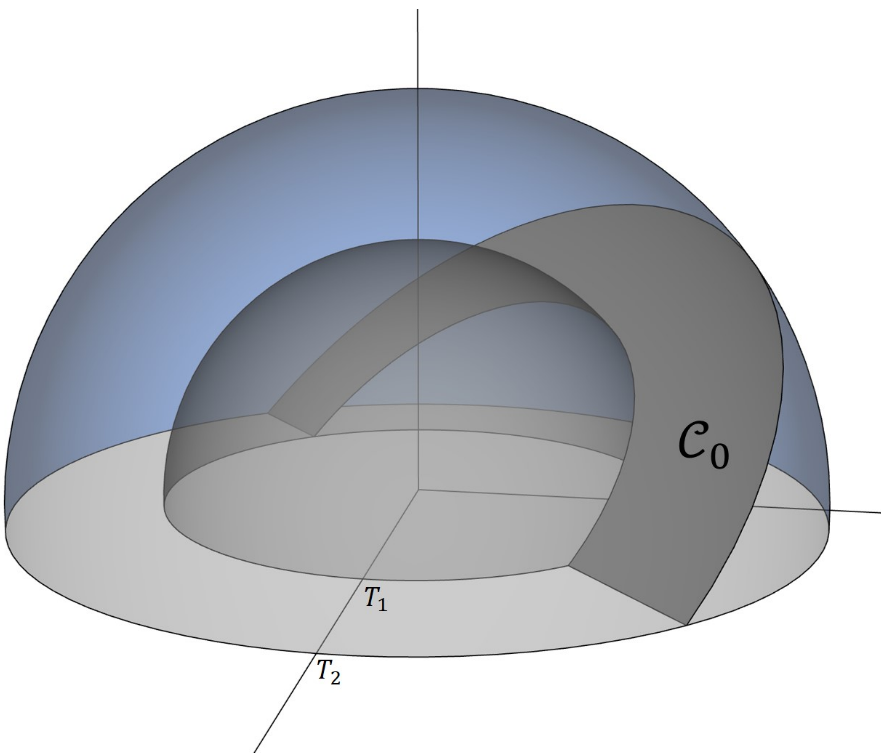


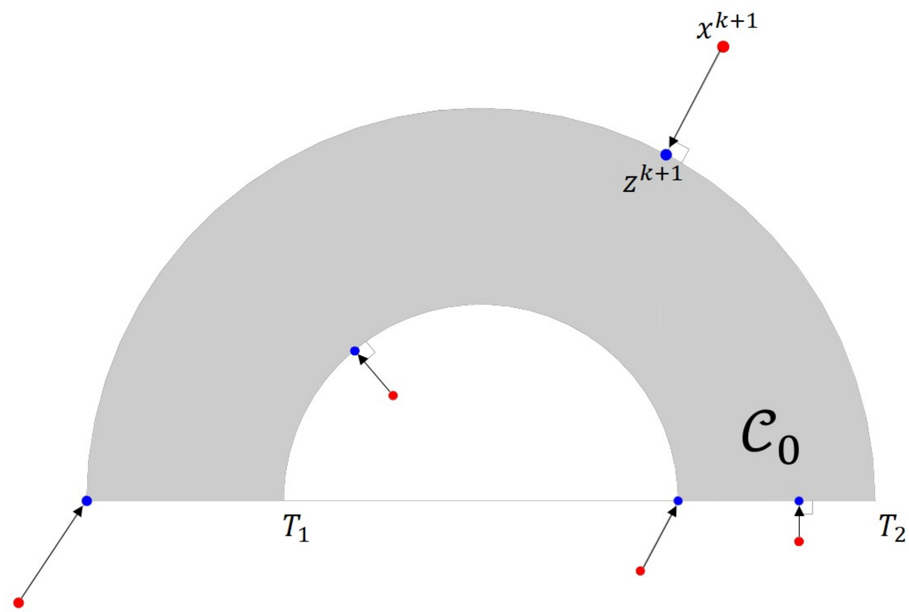
Examples

Problems with thrust bounds:

$$0 \leq \rho_1 \leq \|u_t\| \leq \rho_2, \qquad u_{t,3} \geq 0$$

Projection onto a hemispherical shell:



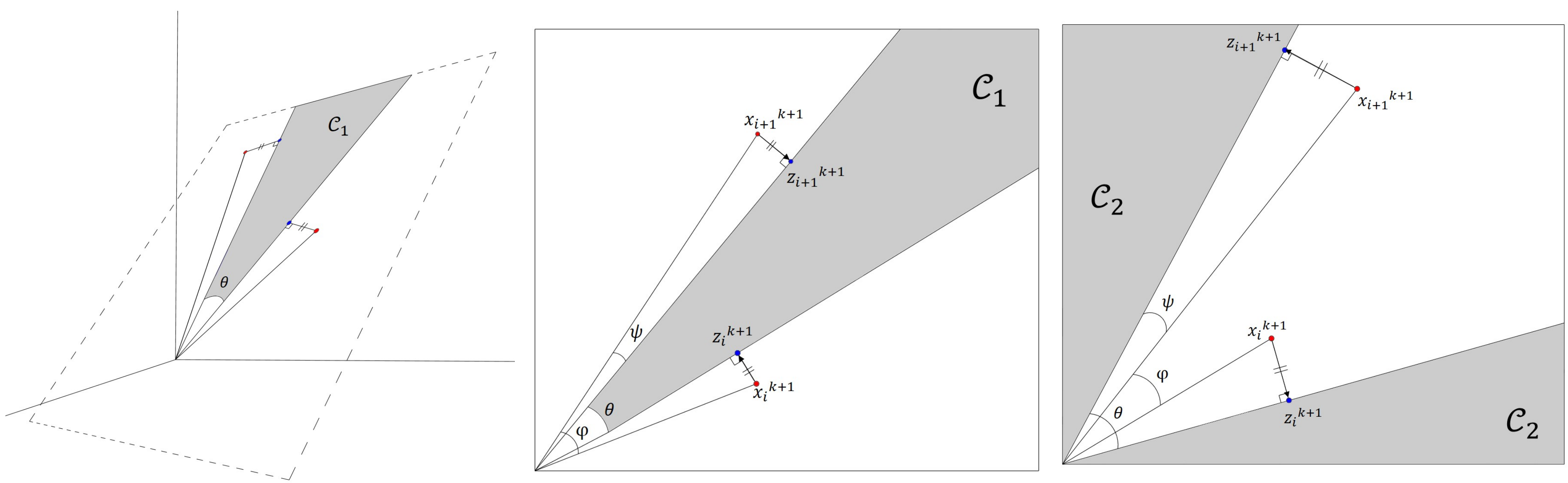


Examples

Problems with angle constraints between state vectors:

$$\angle(v_t, u_t) \leq \theta \qquad \text{or} \qquad \angle(v_t, u_t) \geq \theta$$

Projection around a state dependent cone:



Concluding remarks

Optimal control under nonconvex constraints:

- Some with interesting nonconvex constraints can be handled:
  - via applying the ADMM with direct projection operations onto the nonconvex set
  - no approximation/relaxation required
- Convergence and optimality:
  - converges to very good solutions in practice
  - global optimality not known in general
  - however some results on convergence and global optimality are being reported recently
- Computational complexity:
  - almost equal to that of a single convex optimization problem