ASE2010 Applied linear algebra: Homework #3

1) *Rotation matrices.* Consider a matrix A that describes a rotation by θ, that is,

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

a) Explain why $\|y\| = \|x\|$ for any x and θ.

b) Show that the columns of A are orthonormal vectors.

c) Construct a matrix that describes a rotation by $-\theta$?

d) What is A^T? Is it equal to what you obtained from above?

e) Consider a vector x, and suppose that we compute $y = Ax$, and then subsequently compute $z = A^T y$. What is z?

f) What is $A + A^T$? What does it do? Justify your answer by drawing a picture on a plane to illustrate x, Ax, $A^T x$, and $(A + A^T)x$

2) *Quadratic form.* Suppose P is an $n \times n$ matrix. The function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = x^T P x$ is called a *quadratic form*, and generalizes the idea of a quadratic function of a scalar variable, px^2. The matrix P is called the coefficient matrix of the quadratic form.

a) Show that $f(x) = \sum_{i,j} P_{ij} x_i x_j$. In words: $f(x)$ is the weighted sum of all products of two components of x, with weights given by the entries of P.

b) Show that for any x, we also have $f(x) = x^T P^T x$. In other words, the quadratic form associated with the transpose matrix is the same function.

c) Show that f can be expressed as $f(x) = x^T P^{\mathrm{s}} x$, where $P^{\mathrm{s}} = (1/2)(P + P^T)$ is the symmetric part of P. The matrix P^{s} is symmetric. So any quadratic form can be expressed as one with a coefficient matrix that is symmetric.

d) Express $f(x) = -2x_1^2 + 4x_1x_2 + 2x^2$ in the form $f(x) = x^T P x$ with P a symmetric 2×2 matrix.

e) Suppose that A is an $m \times n$ matrix and b is an m-vector. Show that $\|Ax - b\|^2 = x^T P x + q^T x + r$ for a suitable $n \times n$ symmetric matrix P, n-vector q, and constant r. (Give P , q, and r.) In words: The norm squared of an affine function of x can be expressed as the sum of a quadratic form and an affine function.

3) *VMLS Exercises.*

   a) **7.1** *Projection on a line.*

   b) **7.2** *3-D rotation.*

   c) **7.3** *Trimming a vector.*

   d) **7.4** *Down-sampling and up-conversion.*