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Vector Mechanics For Engineers: Dynamics

Twelfth Edition

Chapter 17

Plane Motion of Rigid Bodies: Energy and Momentum Methods

Contents

Introduction

A Charpy impact test is used to determine the amount of energy absorbed by a material during impact.

To determine the amount of energy absorbed, the final gravitational potential energy of the arm is subtracted from its initial gravitational potential energy.

Introduction

- Method of work and energy and the method of impulse and momentum will be used to analyze the plane motion of rigid bodies and systems of rigid bodies.

- Principle of work and energy is well suited to the solution of problems involving displacements and velocities.

$$T_1 + U_{1\to 2} = T_2$$

- Principle of impulse and momentum is appropriate for problems involving velocities and time.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2 \qquad \left(\vec{H}_O\right)_1 + \sum \int_{t_1}^{t_2} \vec{M}_O dt = \left(\vec{H}_O\right)_2$$

- Problems involving eccentric impact are solved by supplementing the principle of impulse and momentum with the application of the coefficient of restitution.

Introduction 3

Approaches to Rigid Body Kinetics Problem

Forces and Accelerations	Velocities and Displacements	Velocities and Time
Newton's Second Law (last chapter)	**Work-Energy**	**Impulse-Momentum**
$\sum \vec{F} = m\vec{a}_G$ $\sum \vec{M}_G = \dot{\vec{H}}_G$	$T_1 + U_{1\rightarrow 2} = T_2$	$m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2$ $I_G\omega_1 + \int_{t_1}^{t_2} M_G\,dt = I_G\omega_2$

Principle of Work and Energy

Work and kinetic energy are scalar quantities.

- Assume that the rigid body is made of a large number of particles.

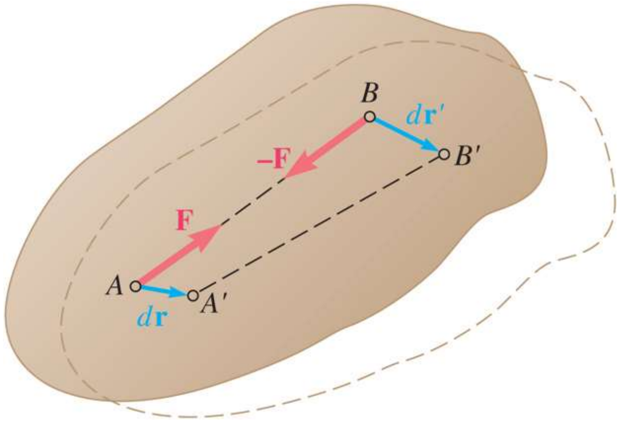
$$T_1 + U_{1\to 2} = T_2$$

$T_1, T_2 =$ initial and final total kinetic energy of particles forming body

$U_{1\to 2} =$ total work of internal and external forces acting on particles of body.

Internal forces between particles *A* and *B* are equal and opposite.

Therefore, *the net work of internal forces is zero.*

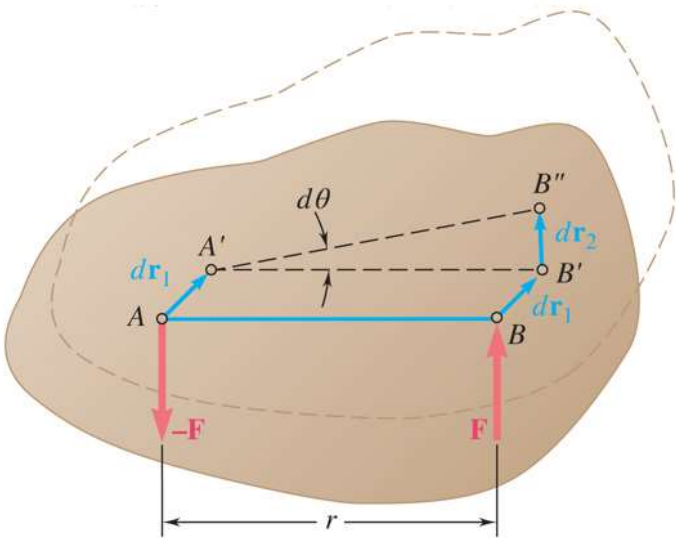


Work of Forces Acting on a Rigid Body 1

- Work of a force during a displacement of its point of application,

$$U_{1\to 2}=\int_{A_1}^{A_2}\vec{F}\cdot d\vec{r}=\int_{s_1}^{s_2}(F\cos\alpha)ds$$

- Consider the net work of two forces $\vec{F}$ and $-\vec{F}$ forming a couple of moment $\vec{M}$ during a displacement of their points of application.



$$\begin{aligned}dU&=\vec{F}\cdot d\vec{r}_1-\vec{F}\cdot d\vec{r}_1+\vec{F}\cdot d\vec{r}_2\\&=F\,ds_2=Fr\,d\theta\\&=M\,d\theta\end{aligned}$$

$$\begin{aligned}U_{1\to 2}&=\int_{\theta_1}^{\theta_2}M\,d\theta\\&=M(\theta_2-\theta_1)\quad\text{if } M \text{ is constant.}\end{aligned}$$

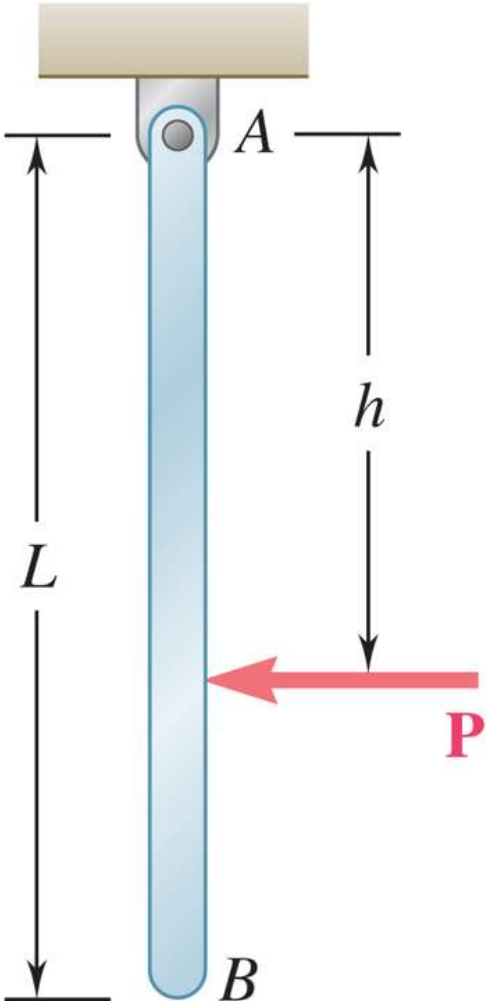
Work of Forces Acting on a Rigid Body 2

Do the pin forces at point A do work?

YES **NO**

Does the force P do work?

YES **NO**



Work of Forces Acting on a Rigid Body

Do the pin forces at point A do work?

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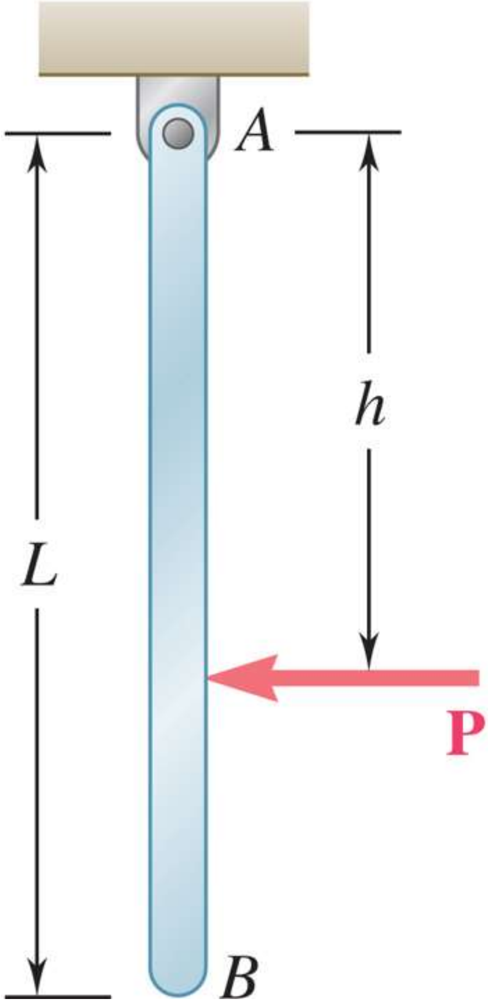
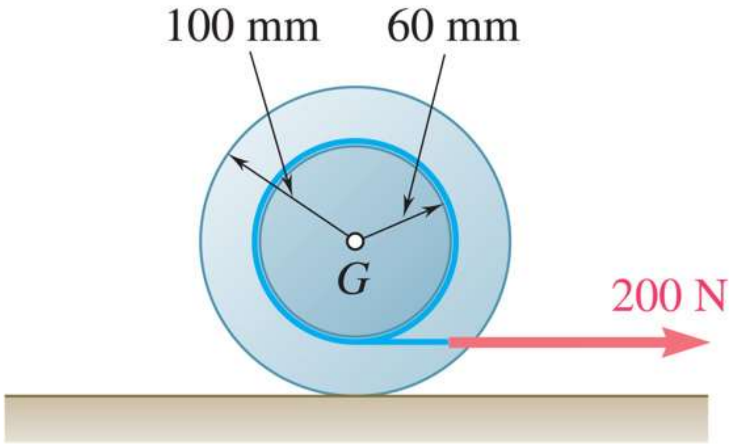
Does the force P do work?

[✓ YES] NO



Work of Forces Acting on a Rigid Body

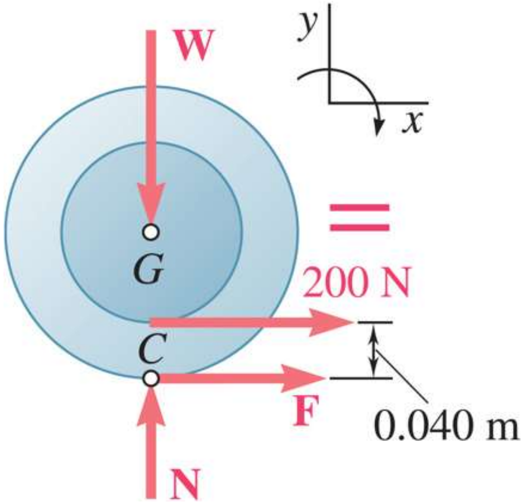


Does the normal force N do work on the disk?

YES **NO**

Does the weight W do work?

YES **NO**



If the disk rolls without slip, does the friction force F do work?

YES **NO**

$$dU = F\,ds_C = F(v_c dt) = 0$$

Work of Forces Acting on a Rigid Body

Does the normal force N do work on the disk?

YES [✓ NO]

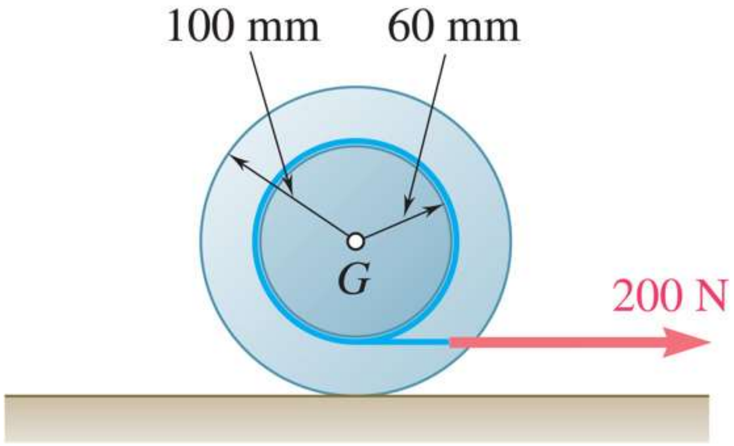
Does the weight W do work?

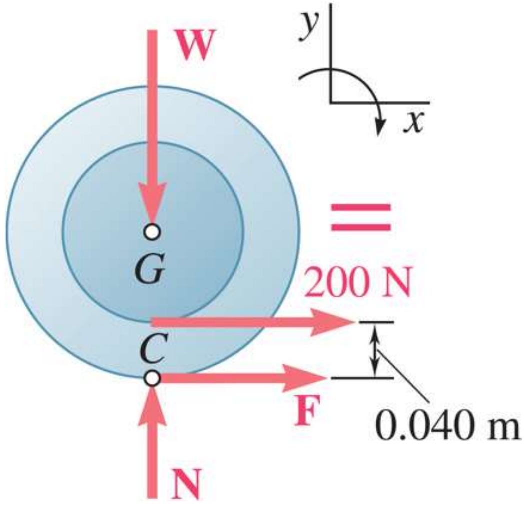
YES [✓ NO]

If the disk rolls without slip, does the friction force F do work?

YES [✓ NO]

$$dU = F\,ds_C = F\left(v_c dt\right) = 0$$

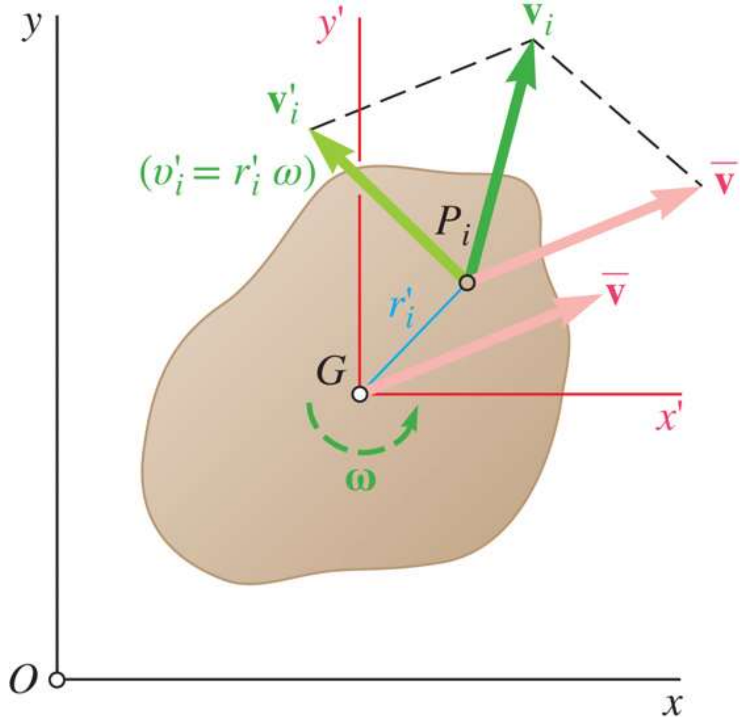




Kinetic Energy of a Rigid Body in Plane Motion 1

Consider a rigid body of mass m in plane motion consisting of individual particles i. The kinetic energy of the body can then be expressed as:

$$\begin{aligned} T &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum \Delta m_i v_i'^2 \\ &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\left(\sum r_i'^2 \Delta m_i\right)\omega^2 \\ &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 \end{aligned}$$



Kinetic energy of a rigid body can be separated into:

- the kinetic energy associated with the motion of the mass center G and
- the kinetic energy associated with the rotation of the body about G.

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$

Translation + Rotation

Kinetic Energy of a Rigid Body in Plane Motion 2

- Consider a rigid body rotating about a fixed axis through O.

$$T = \tfrac{1}{2}\sum \Delta m_i v_i^2 = \tfrac{1}{2}\sum \Delta m_i \left(r_i\omega\right)^2 = \tfrac{1}{2}\left(\sum r_i^2 \Delta m_i\right)\omega^2$$

$$= \tfrac{1}{2} I_O \omega^2$$

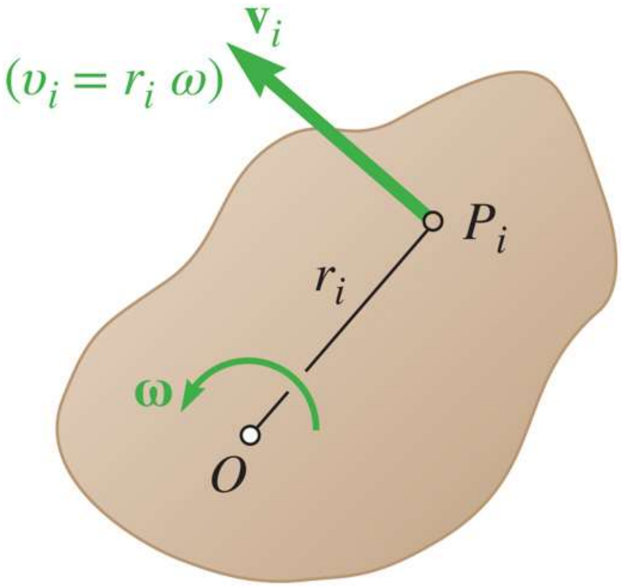
- This is equivalent to using:

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$

- Remember to only use

$$T = \tfrac{1}{2} I_O \omega^2$$

when O is a fixed axis of rotation



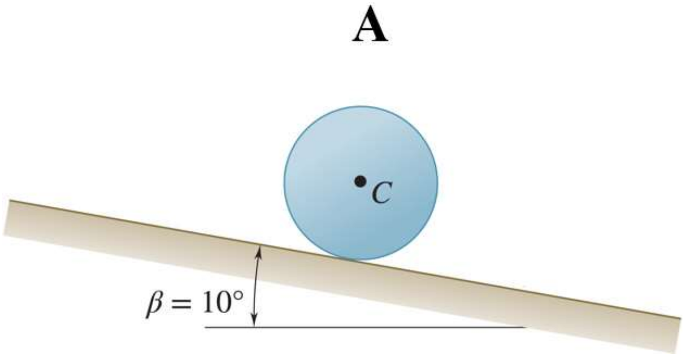
Concept Quiz

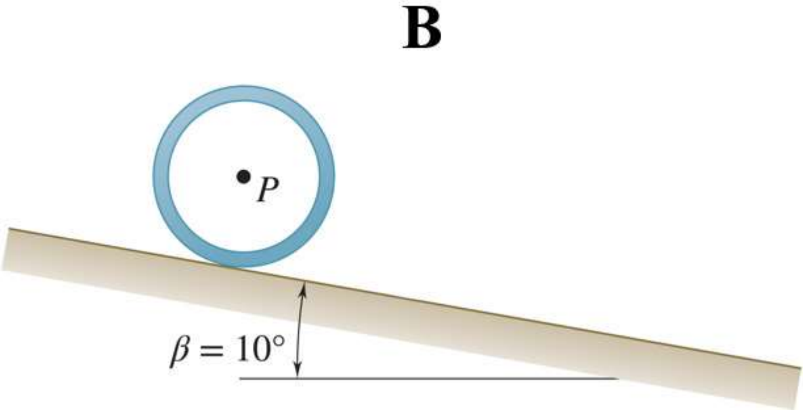
The solid cylinder A and the pipe B have the same diameter and mass. If they are both released from rest at the top of the hill, which will reach the bottom the fastest?

a) A will reach the bottom first

b) B will reach the bottom first

c) They will reach the bottom at the same time



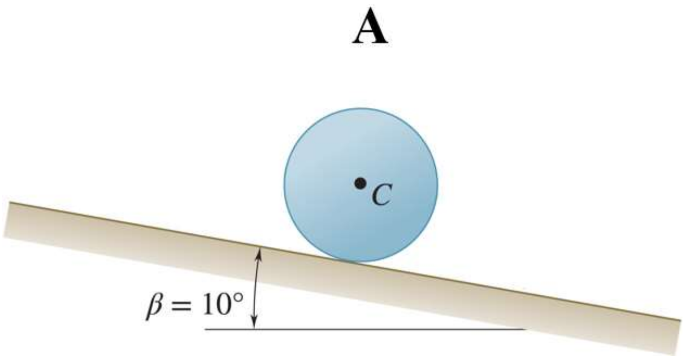


Which will have the greatest kinetic energy when it reaches the bottom?

a) Cylinder A   b) Pipe B   c) Same kinetic energy

Concept Quiz 2

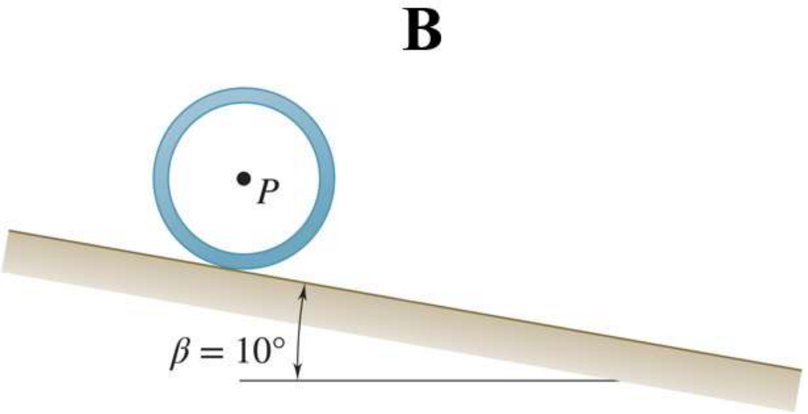
The solid cylinder A and the pipe B have the same diameter and mass. If they are both released from rest at the top of the hill, which will reach the bottom the fastest?



[✓ a) A will reach the bottom first]

b) B will reach the bottom first

c) They will reach the bottom at the same time



Which will have the greatest kinetic energy when it reaches the bottom?

a) Cylinder A b) Pipe B [✓ c) Same kinetic energy]

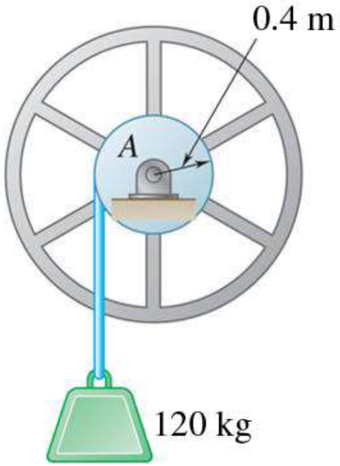
Systems of Rigid Bodies 1

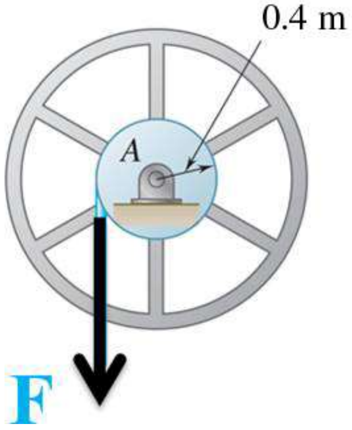
- For problems involving systems consisting of several rigid bodies, the principle of work and energy can be applied to each body.
- We may also apply the principle of work and energy to the entire system,

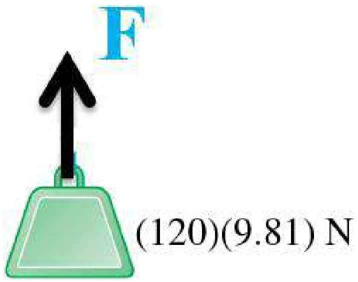
$$T_1 + U_{1\to 2} = T_2$$

T_1, T_2 = arithmetic sum of the kinetic energies of all bodies forming the system

$U_{1\to 2}$ = work of all forces acting on the various bodies, whether these forces are internal or external to the system as a whole.



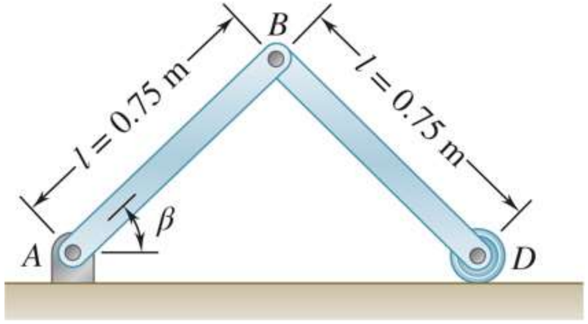


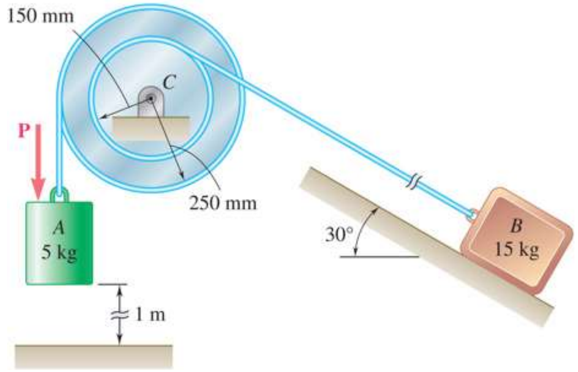


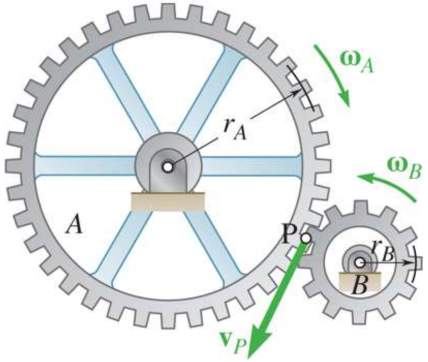
Systems of Rigid Bodies 2

For problems involving pin connected members, blocks and pulleys connected by inextensible cords, and meshed gears,

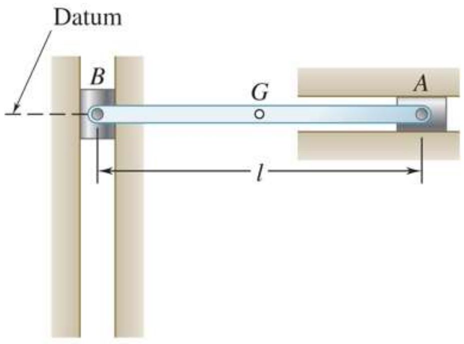
- internal forces occur in pairs of equal and opposite forces
- points of application of each pair move through equal distances
- net work of the internal forces is zero
- work on the system reduces to the work of the external forces



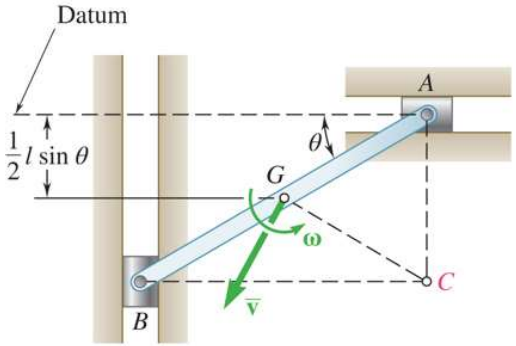




Conservation of Energy



(a)



(b)

- mass *m*
- released with zero velocity
- determine w at q

- Expressing the work of conservative forces as a change in potential energy, the principle of work and energy becomes

$$T_1 + V_1 = T_2 + V_2$$

- Consider the slender rod of mass *m*.

$$T_1 = 0, \quad V_1 = 0$$

$$T_2 = \tfrac{1}{2} m \bar{v}_2^2 + \tfrac{1}{2} \bar{I} \omega_2^2$$

$$= \tfrac{1}{2} m \left(\tfrac{1}{2} l \omega\right)^2 + \tfrac{1}{2} \left(\tfrac{1}{12} m l^2\right) \omega^2 = \frac{1}{2} \frac{m l^2}{3} \omega^2$$

$$V_2 = -\tfrac{1}{2} W l \sin\theta = -\tfrac{1}{2} m g l \sin\theta$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 = \frac{1}{2} \frac{m l^2}{3} \omega^2 - \frac{1}{2} m g l \sin\theta$$

$$\omega = \left(\frac{3g}{l} \sin\theta \right)$$

Power

- Power = rate at which work is done

- For a body acted upon by force $\vec{F}$ and moving with velocity $\vec{v}$,

$$\text{Power} = \frac{dU}{dt} = \vec{F} \cdot \vec{v}$$

- For a rigid body rotating with an angular velocity $\vec{\omega}$ and acted upon by a couple of moment $\vec{M}$ parallel to the axis of rotation,

$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega$$

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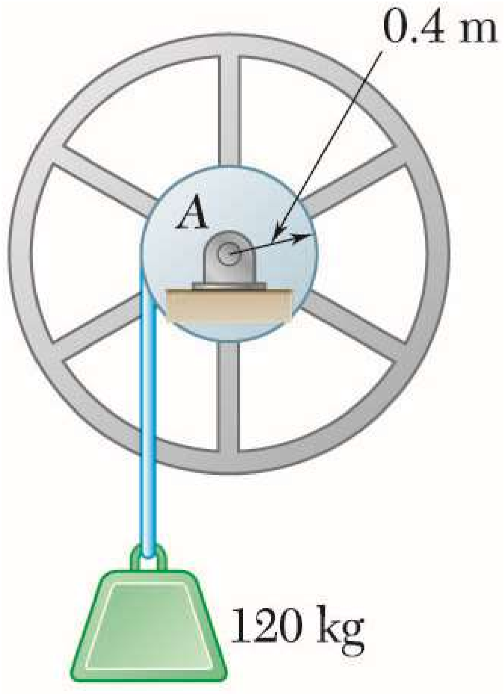
Sample Problem 17.1



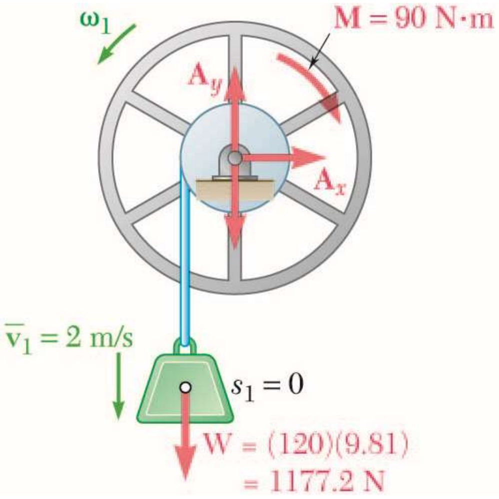
For the drum and flywheel, $\bar{I} = 16\text{kg} \cdot \text{m}^2$. The bearing friction is equivalent to a couple of $90\text{N} \cdot \text{m}$. At the instant shown, the block is moving downward at 2 m/s.

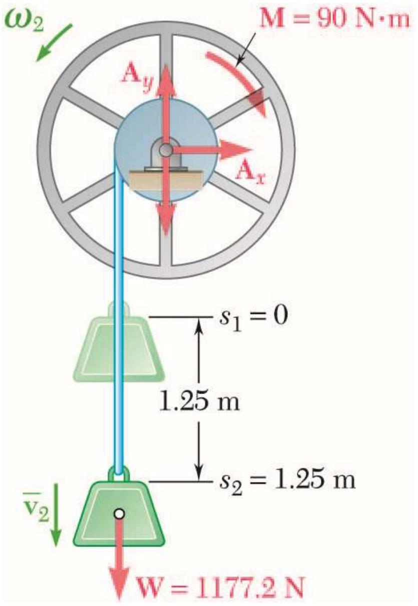
Determine the velocity of the block after it has moved 1.25 m downward.

Strategy:

- Consider the system of the flywheel and block. The work done by the internal forces exerted by the cable cancels.
- Note that the velocity of the block and the angular velocity of the drum and flywheel are related by
  $$\bar{v} = r\omega$$
- Apply the principle of work and kinetic energy to develop an expression for the final velocity.

Sample Problem 17.1 2





Modeling and Analysis:

- Consider the system of the flywheel and block. The work done by the internal forces exerted by the cable cancels.
- Note that the velocity of the block and the angular velocity of the drum and flywheel are related by

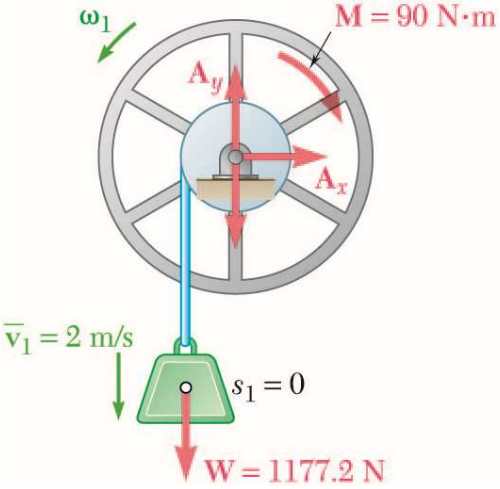
$$\bar{v} = r\omega \qquad \omega_1 = \frac{\bar{v}_1}{r} = \frac{2\text{m/s}}{0.4\text{m}} = 5\,\frac{\text{rad}}{\text{s}} \qquad \omega_2 = \frac{\bar{v}_2}{r} = \frac{\bar{v}_2}{0.4}$$

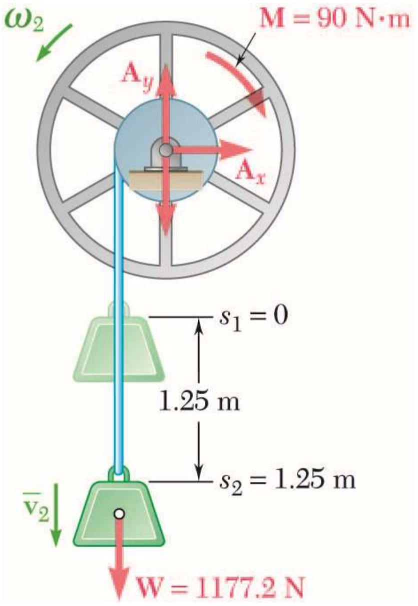
- Apply the principle of work and kinetic energy to develop an expression for the final velocity.

$$\begin{aligned} T_1 &= \frac{1}{2}mv_1^2 + \frac{1}{2}\bar{I}\omega_1^2 \\ &= \frac{1}{2}(120\ \text{kg})\left(2\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(16\text{kg}\cdot\text{m}^2)\left(5\,\frac{\text{rad}}{\text{s}}\right)^2 \\ &= 440\text{J} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}(120\ \text{kg})\bar{v}_2^2 + \frac{1}{2}16\left(\frac{v_2}{0.4}\right)^2 = 110\bar{v}_2^2 \end{aligned}$$

Sample Problem 17.1





$$T_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}\bar{I}\omega_1^2 = 440\text{J}$$

$$T_2 = \frac{1}{2}m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 = 110\bar{v}_2^2$$

- Note that the block displacement and pulley rotation are related by

$$\theta_2 = \frac{s_2}{r} = \frac{1.25\text{m}}{0.4\text{m}} = 3.125\text{rad}$$

Then,

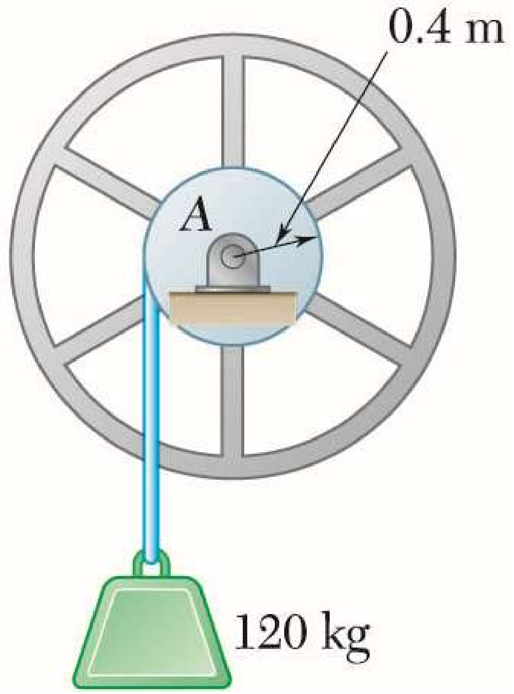
$$\begin{aligned} U_{1\to 2} &= W(s_2 - s_1) - M(\theta_2 - \theta_1) \\ &= (1177 \quad \cdot \quad 2\text{N})(1.25\text{m}) - (90\text{N}\cdot\text{m})(3.125\text{rad}) \\ &= 1190\text{J} \end{aligned}$$

- Principle of work and energy:

$$\begin{aligned} &T_1 + U_{1\to 2} = T_2 \\ &440\text{J} + 1190\text{J} = 110\bar{v}_2^2 \\ &\bar{v}_2 = 3.85\frac{\text{m}}{\text{s}} \end{aligned}$$

$$\boxed{\bar{v}_2 = 3.85\,\text{m/s} \downarrow}$$

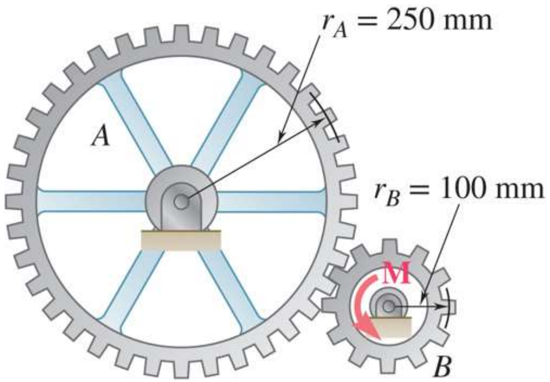
Sample Problem 17.1



Reflect and Think:

- The speed of the block increases as it falls, but much more slowly than if it were in free fall. This seems like a reasonable result.
- Rather than calculating the work done by gravity, you could have also treated the effect of the weight using gravitational potential energy, V_g.

Sample Problem 17.2 1



$m_A = 10\text{kg}$ $\bar{k}_A = 200\text{mm}$

$m_B = 3\text{kg}$ $\bar{k}_B = 80\text{mm}$

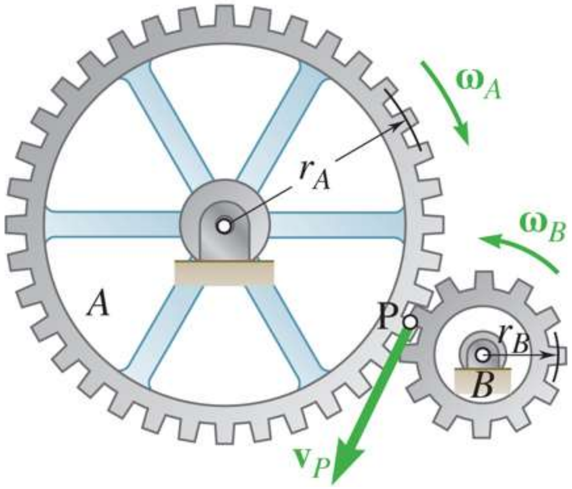
The system is at rest when a moment of M=6 N · m is applied to gear *B*.

Neglecting friction, *a*) determine the number of revolutions of gear *B* before its angular velocity reaches 600 rpm, and *b*) tangential force exerted by gear *B* on gear *A*.

Strategy:

- Consider a system consisting of the two gears. Noting that the gear rotational speeds are related, evaluate the final kinetic energy of the system.
- Apply the principle of work and energy. Calculate the number of revolutions required for the work of the applied moment to equal the final kinetic energy of the system.
- Apply the principle of work and energy to a system consisting of gear *A*. With the final kinetic energy and number of revolutions known, calculate the moment and tangential force required for the indicated work.

Sample Problem 17.2 2



Modeling and Analysis:

- Consider a system consisting of the two gears. Noting that the gear rotational speeds are related, evaluate the final kinetic energy of the system.

$$\omega_B = \frac{(600\,\text{rpm})(2\pi\,\text{rad/rev})}{60\,\text{s/min}} = 62.8\,\text{rad/s}$$

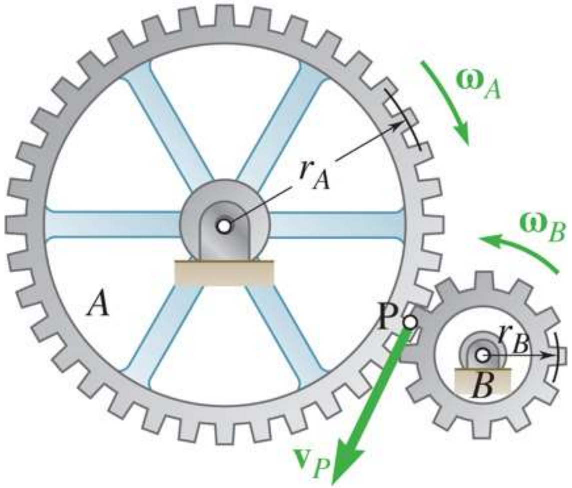
$$\omega_A = \omega_B \frac{r_B}{r_A} = 62.8 \frac{0.100}{0.250} = 25.1\,\text{rad/s}$$

$$\bar{I}_A = m_A \bar{k}_A^2 = (10\text{kg})(0.200\text{m})^2 = 0.400\,\text{kg}\cdot\text{m}^2$$

$$\bar{I}_B = m_B \bar{k}_B^2 = (3\text{kg})(0.080\text{m})^2 = 0.0192\,\text{kg}\cdot\text{m}^2$$

$$\begin{aligned} T_2 &= \tfrac{1}{2}\bar{I}_A\omega_A^2 + \tfrac{1}{2}\bar{I}_B\omega_B^2 \\ &= \tfrac{1}{2}(0.400)(25.1)^2 + \tfrac{1}{2}(0.0192)(62.8)^2 \\ &= 163.9\,\text{J} \end{aligned}$$

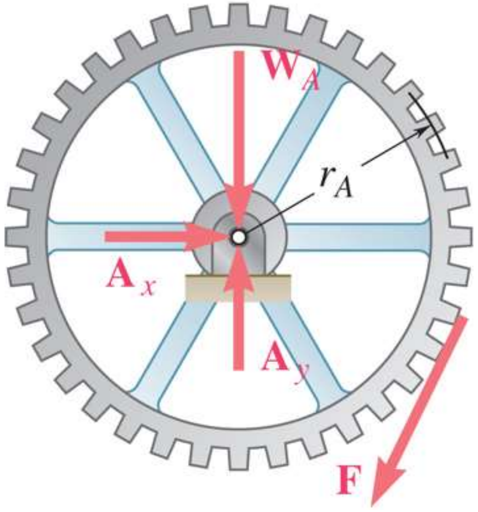
Sample Problem 17.2



- Apply the principle of work and energy. Calculate the number of revolutions required for the work.

$$T_1 + U_{1\to2} = T_2$$
$$0 + (6\theta_B)\mathrm{J} = 163.9\mathrm{J}$$
$$\theta_B = 27.32\,\mathrm{rad}$$

$$\boxed{\theta_B = \frac{27.32}{2\pi} = 4.35\,\mathrm{rev}}$$



- Apply the principle of work and energy to a system consisting of gear *A*. Calculate the moment and tangential force required for the indicated work.

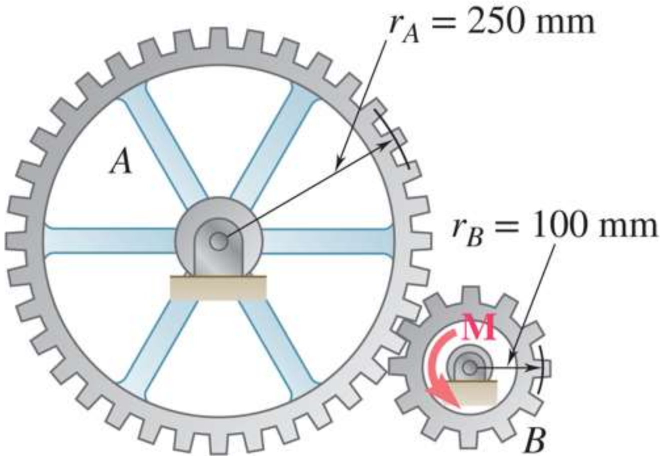
$$\theta_A = \theta_B \frac{r_B}{r_A} = 27.32 \frac{0.100}{0.250} = 10.93\,\mathrm{rad}$$

$$T_2 = \tfrac{1}{2}\bar{I}_A\omega_A^2 = \tfrac{1}{2}(0.400)(25.1)^2 = 126.0\,\mathrm{J}$$

$$T_1 + U_{1\to2} = T_2$$
$$0 + M_A(10.93\,\mathrm{rad}) = 126.0\mathrm{J}$$
$$M_A = r_A F = 11.52\,\mathrm{N\cdot m}$$

$$\boxed{F = \frac{11.52}{0.250} = 46.2\,\mathrm{N}}$$

Sample Problem 17.2



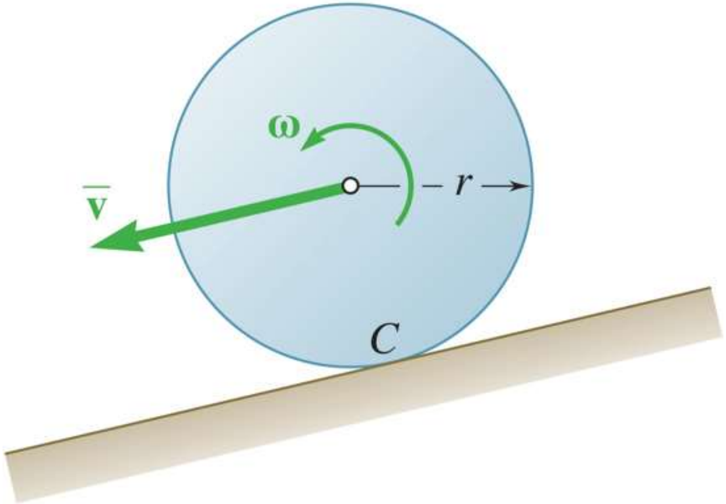
$m_A = 10\,\text{kg} \quad \bar{k}_A = 200\,\text{mm}$

$m_B = 3\,\text{kg} \quad \bar{k}_B = 80\,\text{mm}$

Reflect and Think:

- When the system was both gears, the tangential force between the gears did not appear in the work–energy equation, since it was internal to the system and therefore did no work. If you want to determine an internal force, you need to define a system where the force of interest is an external force. This problem, like most problems, also could have been solved using Newton's second law and kinematic relationships.

Sample Problem 17.3

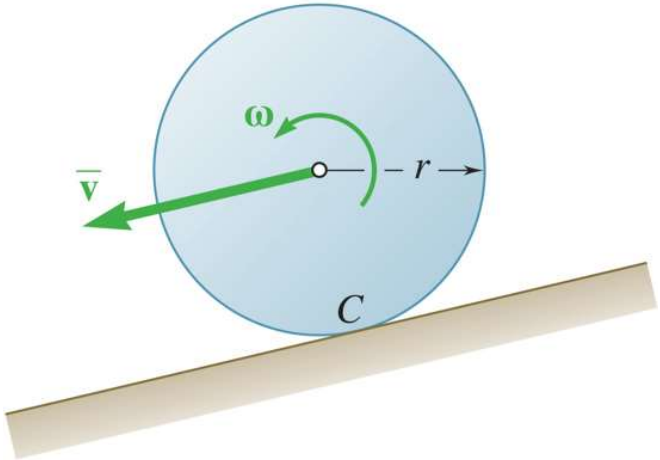


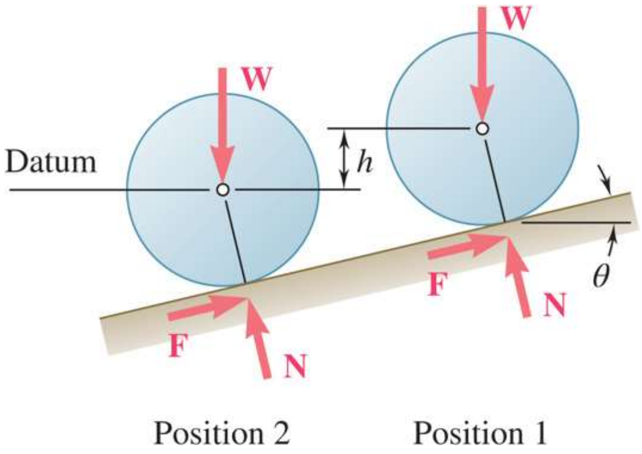
A sphere, cylinder, and hoop, each having the same mass and radius, are released from rest on an incline. Determine the velocity of each body after it has rolled through a distance corresponding to a change of elevation *h*.

Strategy:

- The work done by the weight of the bodies is the same. From the principle of work and energy, it follows that each body will have the same kinetic energy after the change of elevation.
- Because each of the bodies has a different centroidal moment of inertia, the distribution of the total kinetic energy between the linear and rotational components will be different as well.

Sample Problem 17.3 2





Modeling and Analysis:

- The work done by the weight of the bodies is the same. From the principle of work and energy, it follows that each body will have the same kinetic energy after the change of elevation.

With $\omega = \dfrac{\bar{v}}{r}$

$$T_2 = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2$$

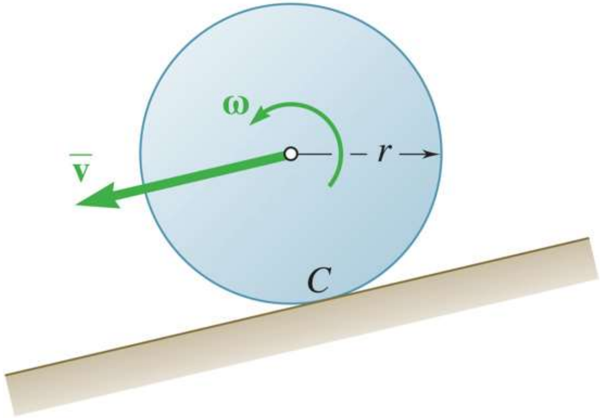
$$= \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

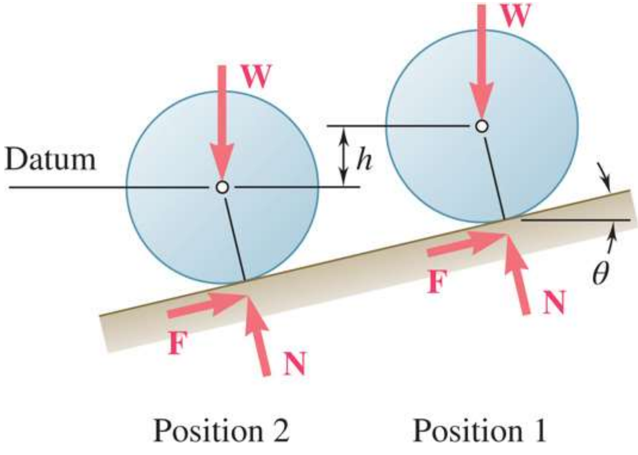
$$T_1 + U_{1\rightarrow 2} = T_2$$

$$0 + Wh = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

$$\bar{v}^2 = \frac{2Wh}{m + \bar{I}/r^2} = \frac{2gh}{1 + \bar{I}/mr^2}$$

Sample Problem 17.3 3





- Because each of the bodies has a different centroidal moment of inertia, the distribution of the total kinetic energy between the linear and rotational components will be different as well.

$$\bar{v}^2 = \frac{2gh}{1 + \bar{I}/mr^2}$$

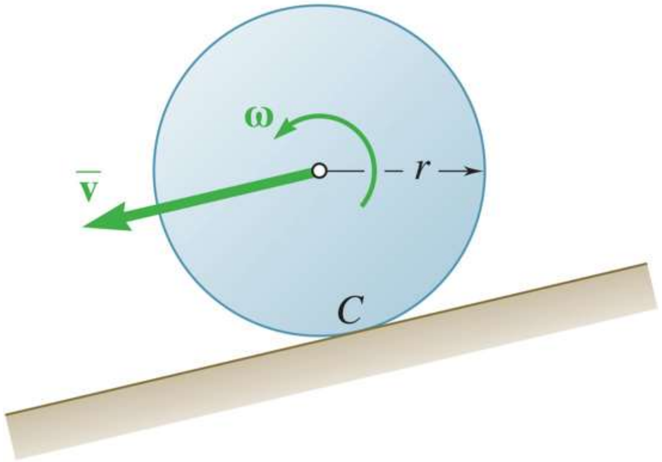
$$\begin{array}{lll} Sphere: & \bar{I} = \tfrac{2}{5}mr^2 & \bar{v} = 0.845\sqrt{2gh} \\ Cylinder: & \bar{I} = \tfrac{1}{2}mr^2 & \bar{v} = 0.816\sqrt{2gh} \\ Hoop: & \bar{I} = mr^2 & \bar{v} = 0.707\sqrt{2gh} \end{array}$$

NOTE:

- For a frictionless block sliding through the same distance, $\omega = 0, \quad \bar{v} = \sqrt{2gh}$
- The velocity of the body is independent of its mass and radius.
- The velocity of the body does depend on

$$\bar{I}\Big/mr^2 = \bar{k}^2\Big/r^2$$

Sample Problem 17.3

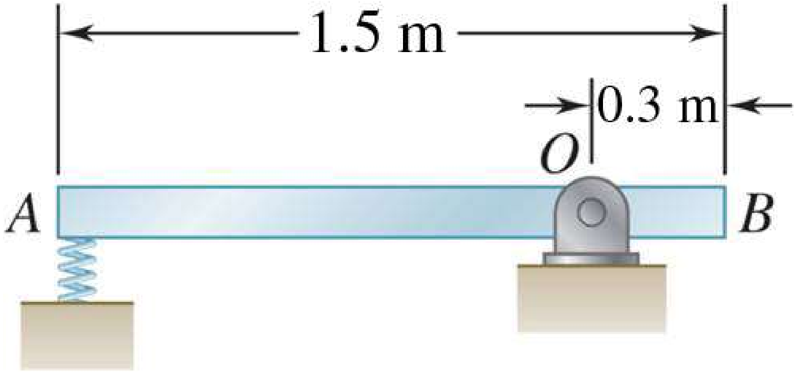


Reflect and Think:

- Let us compare the results with the velocity attained by a frictionless block sliding through the same distance. The solution is identical to the previous solution except that $\omega = 0$; we find $v = \sqrt{2gh}$.

- Comparing the results, we note that the velocity of the body is independent of both its mass and radius. However, the velocity does depend upon the quotient of $I / mr^2 = k^2 / r^2$, which measures the ratio of the rotational kinetic energy to the translational kinetic energy. Thus the hoop, which has the largest k for a given radius r, attains the smallest velocity, whereas the sliding block, which does not rotate, attains the largest velocity.

Sample Problem 17.4 1



A 15-kg slender rod pivots about the point *O*. The other end is pressed against a spring ($k = 300$ kN/m) until the spring is compressed 25 mm and the rod is in a horizontal position.

If the rod is released from this position, determine its angular velocity and the reaction at the pivot as the rod passes through a vertical position.

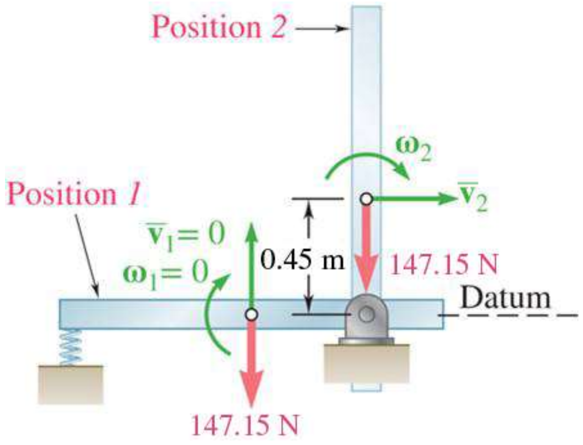
Strategy:

- The weight and spring forces are conservative. The principle of work and energy can be expressed as

$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.
- Express the final kinetic energy in terms of the final angular velocity of the rod.
- Based on the free-body-diagram equation, solve for the reactions at the pivot.

Sample Problem 17.4 2



$$\bar{I} = \frac{1}{12} m l^2$$
$$= \frac{1}{12} (15 \text{ kg})(1.5\text{m})^2$$
$$= 2.8125\text{kg} \cdot \text{m}^2$$

Modeling and Analysis:

- The weight and spring forces are conservative. The principle of work and energy can be expressed as

$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.

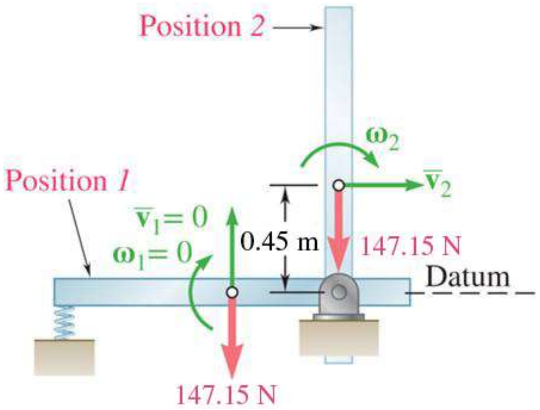
$$V_1 = V_g + V_e = 0 + \frac{1}{2} k x_1^2 = \frac{1}{2} (300{,}000)(0.025\text{m})^2$$
$$= 93.75\text{N} \cdot \text{m}$$

$$V_2 = V_g + V_e = Wh + 0 = (147.15 \text{ N})(0.45 \text{ m})$$
$$= 66.2175 \text{ J}$$

- Express the final kinetic energy in terms of the angular velocity of the rod.

$$T_2 = \frac{1}{2} m \bar{v}_2^2 + \frac{1}{2} \bar{I} \omega_2^2 = \frac{1}{2} m (r\omega_2)^2 + \frac{1}{2} \bar{I} \omega_2^2$$
$$= \frac{1}{2} (15)(0.45\omega_2)^2 + \frac{1}{2} (2.8125)\omega_2^2$$

Sample Problem 17.4 3



From the principle of work and energy,

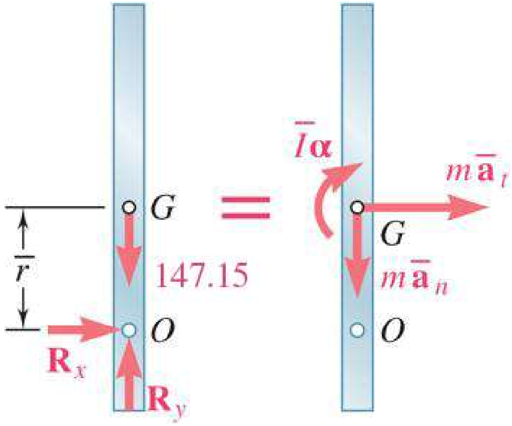
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 93.75\text{J} = 2.925\omega_2^2 + 66.2175\text{J}$$

$$\boxed{\omega_2 = 3.068 \text{ rad/s} \curvearrowright}$$

- Based on the free-body-diagram equation, solve for the reactions at the pivot.

$$\bar{a}_n = \bar{r}\omega_2^2 = (0.45\text{m})(3.068\text{rad/s})^2 = 4.236\text{m/s}^2$$
$$\bar{a}_t = r\alpha$$

$$\vec{\bar{a}}_n = 4.236\text{m/s}^2 \downarrow$$
$$\vec{\bar{a}}_t = r\alpha \rightarrow$$



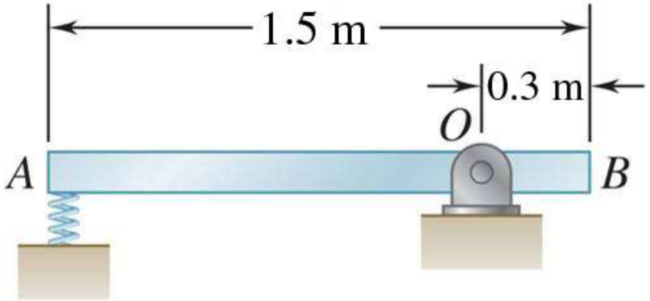
$$+\curvearrowright \ \sum M_O = \sum (M_O)_{eff} \qquad 0 = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} \qquad \alpha = 0$$

$$\overset{+}{\rightarrow} \ \sum F_x = \sum (F_x)_{eff} \qquad R_x = m(\bar{r}\alpha) \qquad R_x = 0$$

$$+\uparrow \ \sum F_y = \sum (F_y)_{eff} \qquad R_y - 147.15\,\text{N} = -ma_n$$
$$= -(15\,\text{kg})(4.236\ \text{m/s}^2)$$
$$R_y = 83.61\,\text{N}$$

$$\boxed{\vec{R} = 83.6\ \text{N} \uparrow}$$

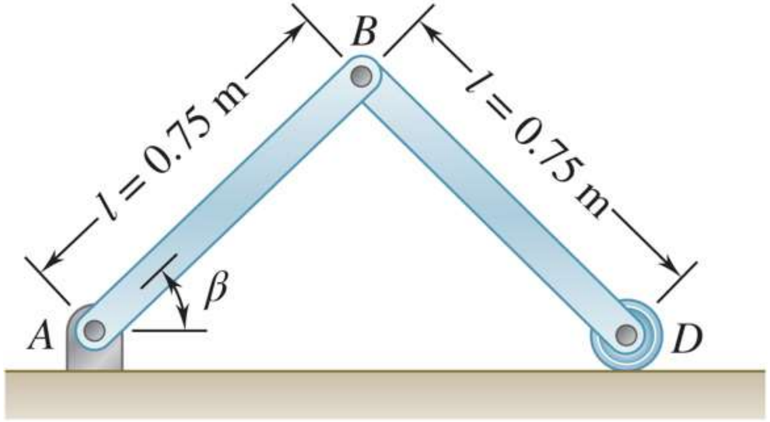
Sample Problem 17.4



Reflect and Think:

- This problem illustrates how you might need to supplement the conservation of energy with Newton’s second law.
- What if the spring constant had been smaller, say 30 kN/m? You would have found V_{e1} = 9.375 J and then solved to obtain $\omega_2^2 = -19.43$.
- This is clearly impossible and means that the rod would not make it to position 2 as assumed.

Sample Problem 17.6



Each of the two slender rods has a mass of 6 kg. The system is released from rest with $\beta = 60^\circ$.

Determine *a*) the angular velocity of rod *AB* when $\beta = 20^\circ$, and *b*) the velocity of the point *D* at the same instant.

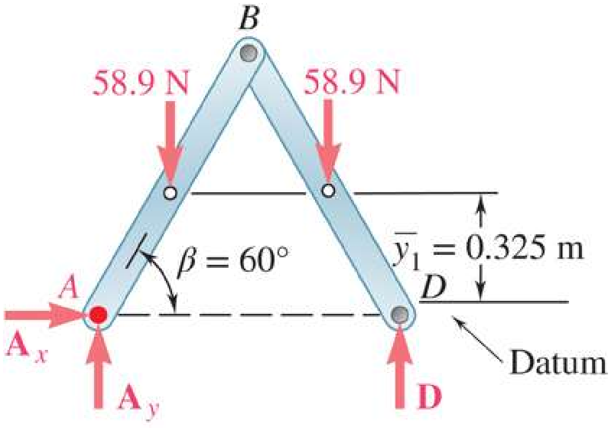
Strategy:

- Consider a system consisting of the two rods. With the conservative weight force,

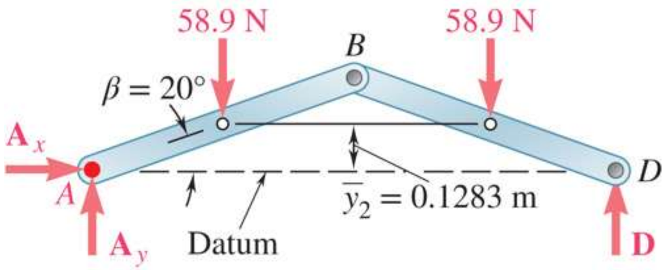
$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.
- Express the final kinetic energy of the system in terms of the angular velocities of the rods.
- Solve the energy equation for the angular velocity, then evaluate the velocity of the point *D*.

Sample Problem 17.6 2



Position 1



Position 2

$$W = mg = (6\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right)$$
$$= 58.86\,\mathrm{N}$$

Modeling and Analysis:

- Consider a system consisting of the two rods. With the conservative weight force,

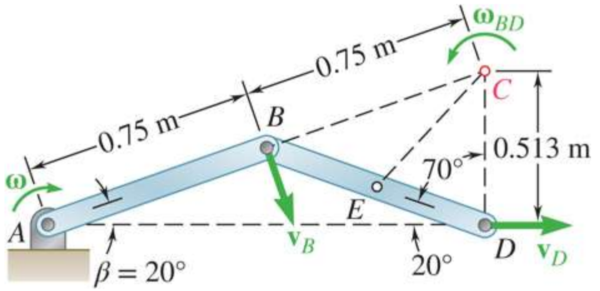
$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.

$$V_1 = 2Wy_1 = 2(58.86\,\mathrm{N})(0.325\,\mathrm{m})$$
$$= 38.26\,\mathrm{J}$$

$$V_2 = 2Wy_2 = 2(58.86\,\mathrm{N})(0.1283\,\mathrm{m})$$
$$= 15.10\,\mathrm{J}$$

Sample Problem 17.6 3



- Express the final kinetic energy of the system in terms of the angular velocities of the rods.

$$\vec{v}_{AB} = (0.375\text{m})\omega \searrow$$

Since $\vec{v}_B$ is perpendicular to *AB* and $\vec{v}_D$ is horizontal, the instantaneous center of rotation for rod *BD* is *C*.

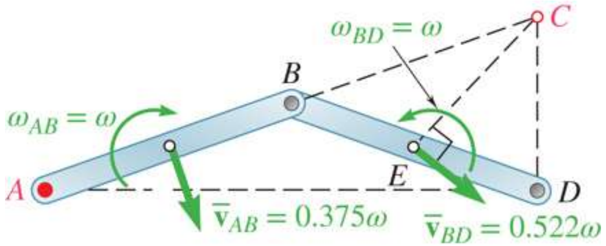
$$\text{BC} = 0.75\text{m} \qquad CD = 2(0.75\,\text{m})\sin 20° = 0.513\,\text{m}$$

and applying the law of cosines to *CDE*, *EC* = 0.522 m

Consider the velocity of point *B*

$$v_B = (AB)\omega = (BC)\omega_{AB} \qquad \vec{\omega}_{BD} = \omega \circlearrowright$$

$$\vec{v}_{BD} = (0.522\text{m})\omega \searrow$$

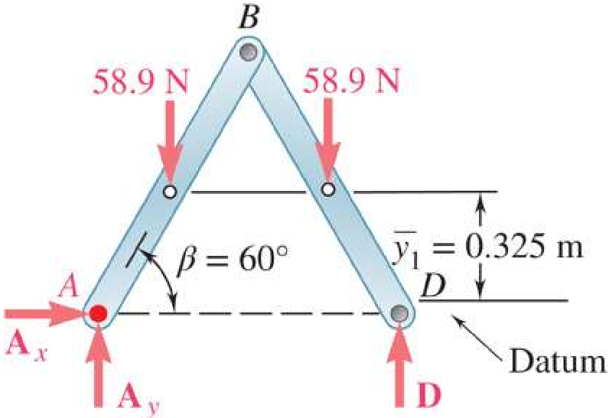


For the final kinetic energy,

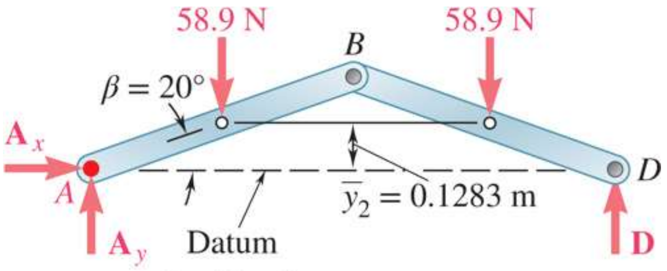
$$\bar{I}_{AB} = \bar{I}_{BD} = \tfrac{1}{12} ml^2 = \tfrac{1}{12}(6\,\text{kg})(0.75\,\text{m})^2 = 0.281\,\text{kg}\cdot\text{m}^2$$

$$\begin{aligned} T_2 &= \tfrac{1}{12} m\bar{v}_{AB}^2 + \tfrac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \tfrac{1}{12} m\bar{v}_{BD}^2 + \tfrac{1}{2}\bar{I}_{BD}\omega_{BD}^2 \\ &= \tfrac{1}{12}(6)(0.375\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 + \tfrac{1}{12}(6)(0.522\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 \\ &= 1.520\omega^2 \end{aligned}$$

Sample Problem 17.6



Position 1



Position 2

- Solve the energy equation for the angular velocity, then evaluate the velocity of the point *D*.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 38.26\,\mathrm{J} = 1.520\omega^2 + 15.10\,\mathrm{J}$$

$$\omega = 3.90\,\mathrm{rad/s}$$

$$\boxed{\vec{\omega}_{AB} = 3.90\,\mathrm{rad/s} \circlearrowright}$$

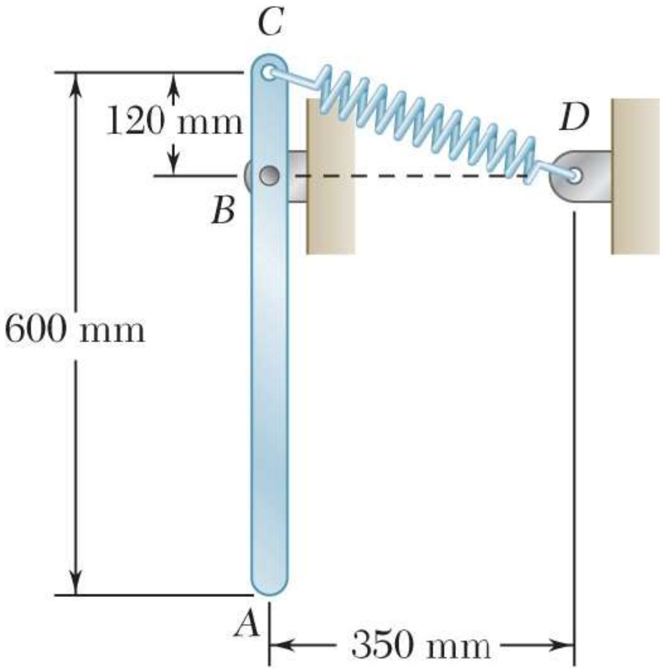
$$\begin{aligned} v_D &= (CD)\omega \\ &= (0.513\,\mathrm{m})(3.90\,\mathrm{rad/s}) \\ &= 2.00\,\mathrm{m/s} \end{aligned}$$

$$\boxed{\vec{\mathrm{v}}_\mathrm{D} = 2.00\,\mathrm{m/s} \rightarrow}$$

Reflect and Think:

The only step in which you need to use forces is when calculating the gravitational potential energy in each position. However, it is good engineering practice to show the complete free-body diagram in each case to identify which, if any, forces do work. You could have also used vector algebra to relate the velocities of the various objects.

Group Problem Solving 1

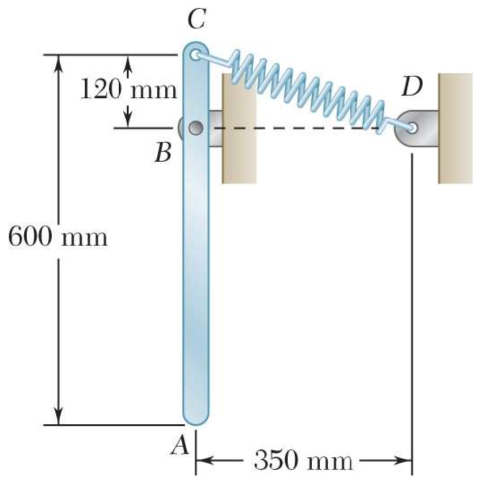


A slender 4-kg rod can rotate in a vertical plane about a pivot at B. A spring of constant k = 400 N/m and of unstretched length 150 mm is attached to the rod as shown. Knowing that the rod is released from rest in the position shown, determine its angular velocity after it has rotated through 90°.

Strategy:

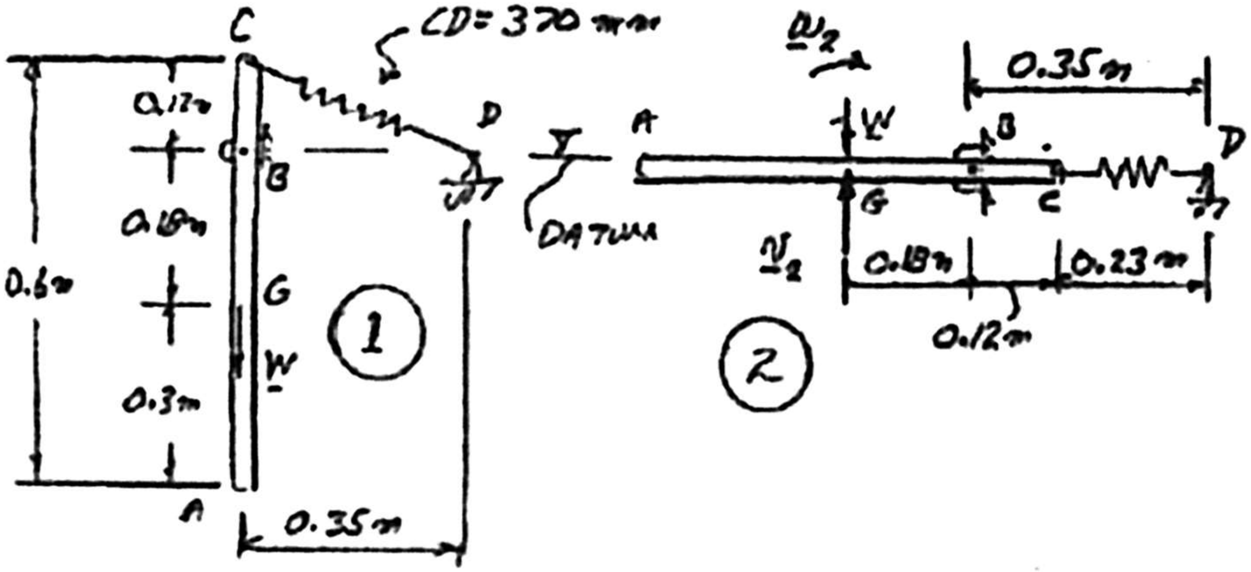
- Because the problem deals with positions and velocities, you should apply the principle of work energy.
- Draw out the system at position 1 and position 2 and define your datum
  - Use the work-energy equation to determine the angular velocity at position 2

Group Problem Solving 2



Modeling and Analysis:
Draw your diagrams, set your datum and apply the work energy equation

$$T_1 + V_1 + U_{1-2} = T_2 + V_2$$



Are any of the terms zero?

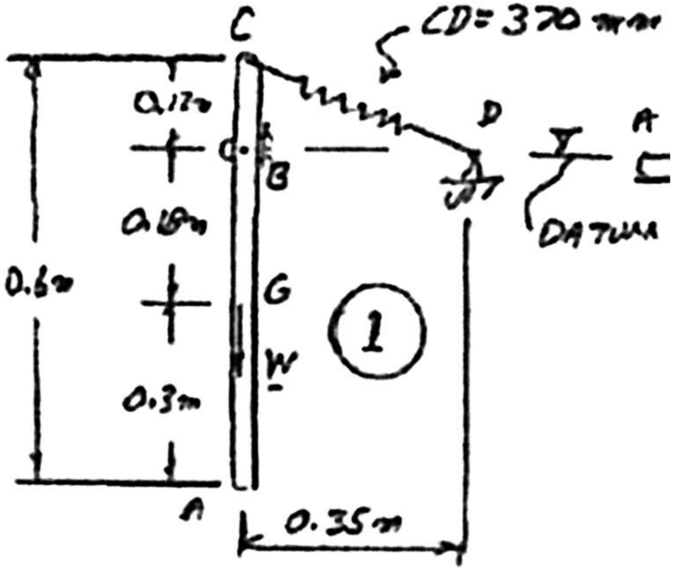
$$\cancel{T_1} + V_1 + \cancel{U_{1-2}} = T_2 + V_2$$

Group Problem Solving 3

Determine the spring energy at position 1

$$x_1 = CD - (\overbrace{150\text{ mm}}^{\text{Unstretched Length}}) = 370 - 150 = 220\text{ mm} = 0.22\text{ m}$$

$$V_e = \frac{1}{2}kx_1^2 = \frac{1}{2}(400\text{ N/m})(0.22\text{ m})^2 = 9.68\text{ J}$$



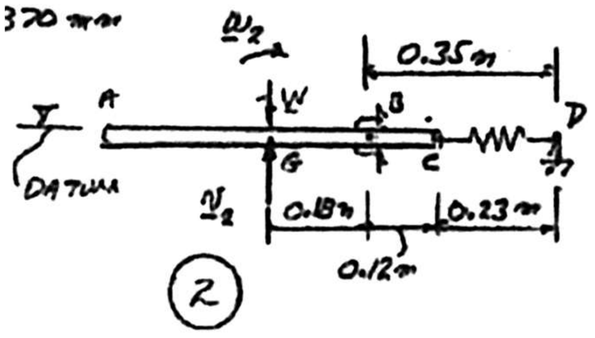
Determine the potential energy due to gravity at position 1

$$V_{g1} = Wh = mgh = (4\text{ kg})(9.81\text{ m/s}^2)(-0.22\text{ m}) = -7.063\text{ J}$$

Determine the spring energy at position 2

$$x_2 = 230\text{ mm} - 150\text{ mm} = 80\text{ mm} = 0.08\text{ m}$$

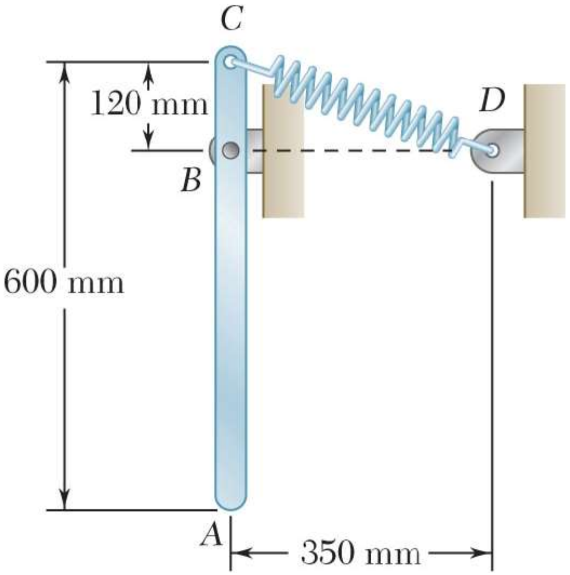
$$V_{e2} = \frac{1}{2}kx_2^2 = \frac{1}{2}(400\text{ N/m})(0.08\text{ m})^2 = 1.28\text{ J}$$



Determine the potential energy due to gravity at position 2

$$V_{g2} = 0$$

Group Problem Solving



Determine an expression for T_2

$$T_2 = \frac{1}{2} m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2$$

Can you relate v_2 and ω_2?

$$\bar{v}_2 = r\omega_2 = (0.18\text{ m})\omega_2$$

Find I and substitute in to T_2

$$\bar{I} = \frac{1}{12} mL^2 = \frac{1}{12}(4\text{ kg})(0.6\text{ m})^2 = 0.12\text{ kg}\cdot\text{m}^2$$

$$T_2 = \frac{1}{2} m\bar{v}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}(4\text{ kg})(0.18\omega_2)^2 + \frac{1}{2}(0.12)\omega_2^2 = 0.1248\omega_2^2$$

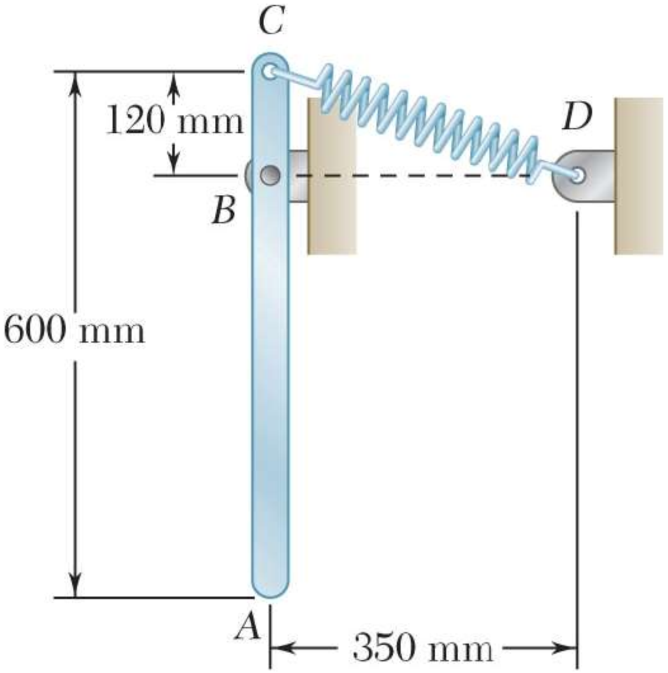
Substitute into $T_1 + V_1 = T_2 + V_2$

$$9.68 - 7.063 = 0.1248\,\omega_2^2 + 1.28\text{ J}$$

$$\omega_2^2 = 10.713$$

$$\boxed{\omega_2 = 3.27\text{ rad/s}}$$

Concept Question 1



For the previous problem, how would you determine the reaction forces at B when the bar is horizontal?

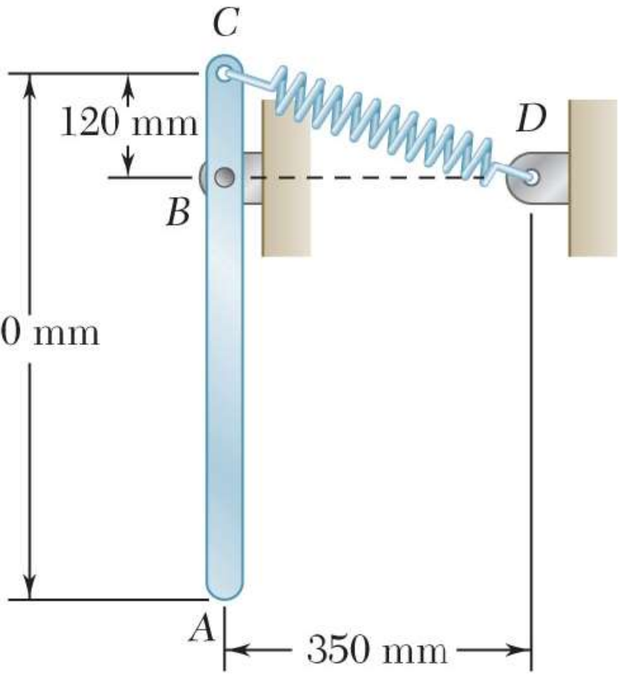
a) Apply linear-momentum to solve for $B_x\Delta t$ and $B_y\Delta t$.

b) Use work-energy to determine the work done by the moment at C.

c) Use sum of forces and sum of moments equations when the bar is horizontal.

Concept Question

For the previous problem, how would you determine the reaction forces at B when the bar is horizontal?



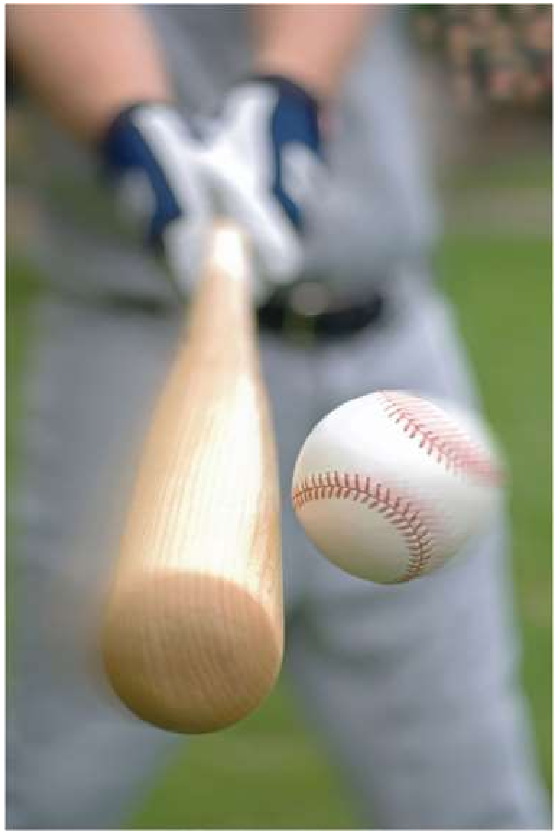
a) Apply linear-momentum to solve for $B_x\Delta t$ and $B_y\Delta t$

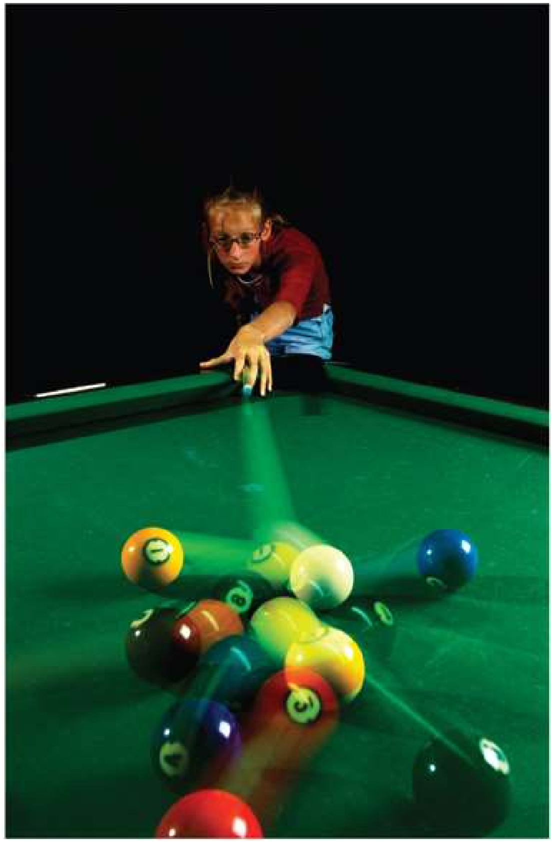
b) Use work-energy to determine the work done by the moment at C

c) Use sum of forces and sum of moments equations when the bar is horizontal

Angular Impulse Momentum

When two rigid bodies collide, we typically use principles of angular impulse momentum. We often also use linear impulse momentum (like we did for particles).





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Introduction

Approaches to Rigid Body Kinetics Problems

Forces and Accelerations

↓

Newton’s Second Law (last chapter)

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

Velocities and Displacements

↓

Work-Energy

$$T_1 + U_{1\to 2} = T_2$$

Velocities and Time

↓

Impulse-Momentum

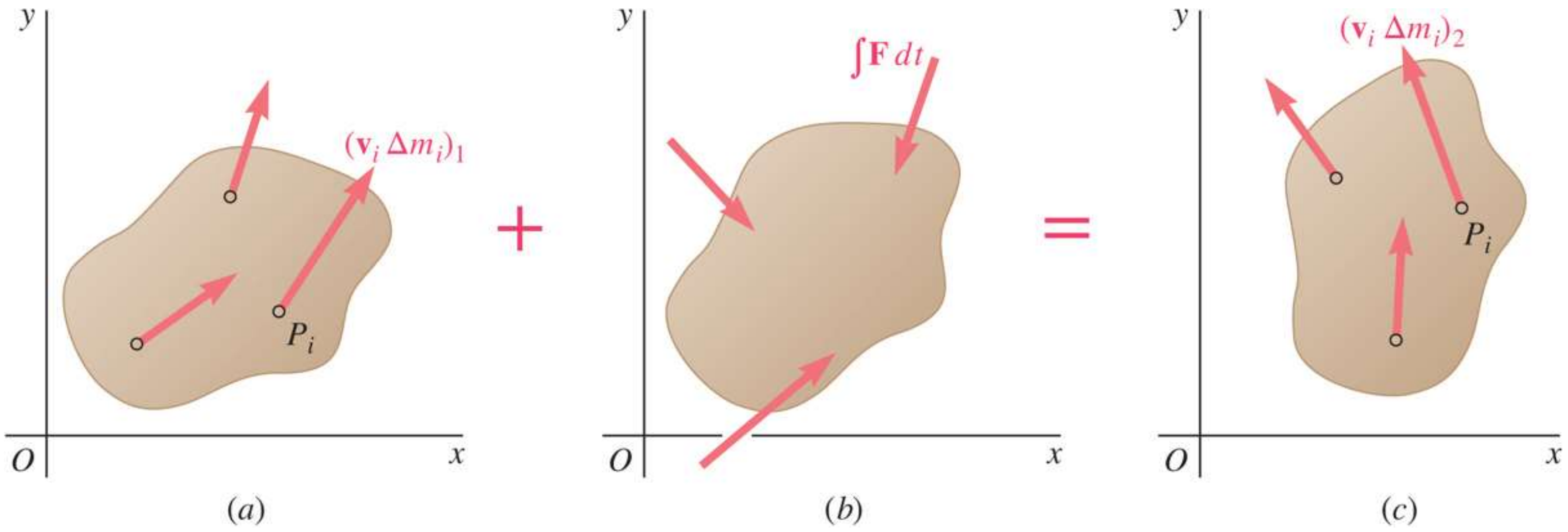
$$m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2$$

$$I_G\omega_1 + \int_{t_1}^{t_2} M_G\,dt = I_G\omega_2$$

Principle of Impulse and Momentum

Method of impulse and momentum:

- well suited to the solution of problems involving time and velocity
- the only practicable method for problems involving impulsive motion and impact.



Sys Momenta$_1$ + Sys Ext Imp$_{1 \text{ to } 2}$ = Sys Momenta$_2$

Principle of Impulse and Momentum 2

- The momenta of the particles of a system may be reduced to a vector attached to the mass center equal to their sum,

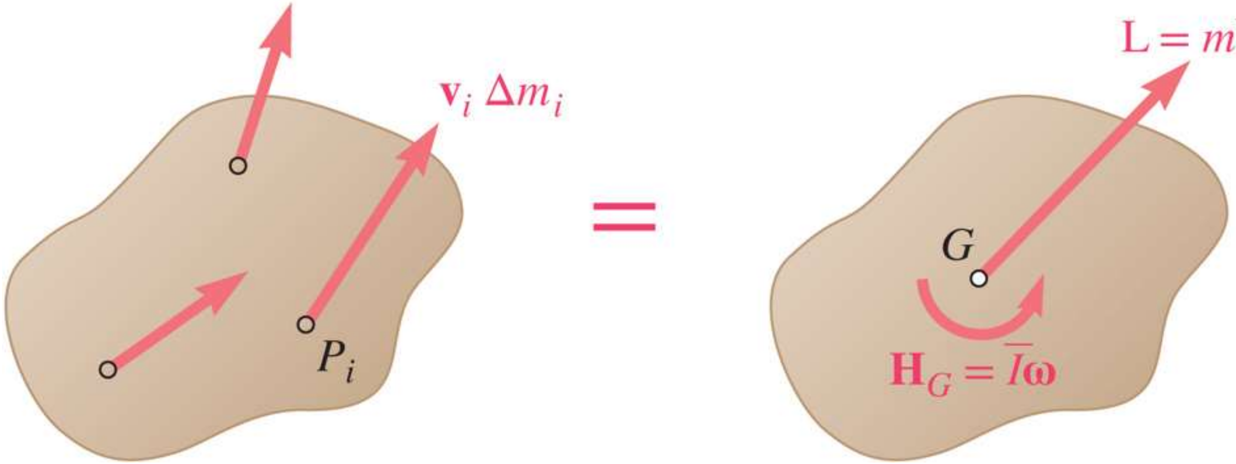
$$\vec{L} = \sum \vec{v}_i \Delta m_i = m\vec{\bar{v}}$$

and a couple equal to the sum of their moments about the mass center,

$$\vec{H}_G = \sum \vec{r}_i' \times \vec{v}_i \Delta m_i$$

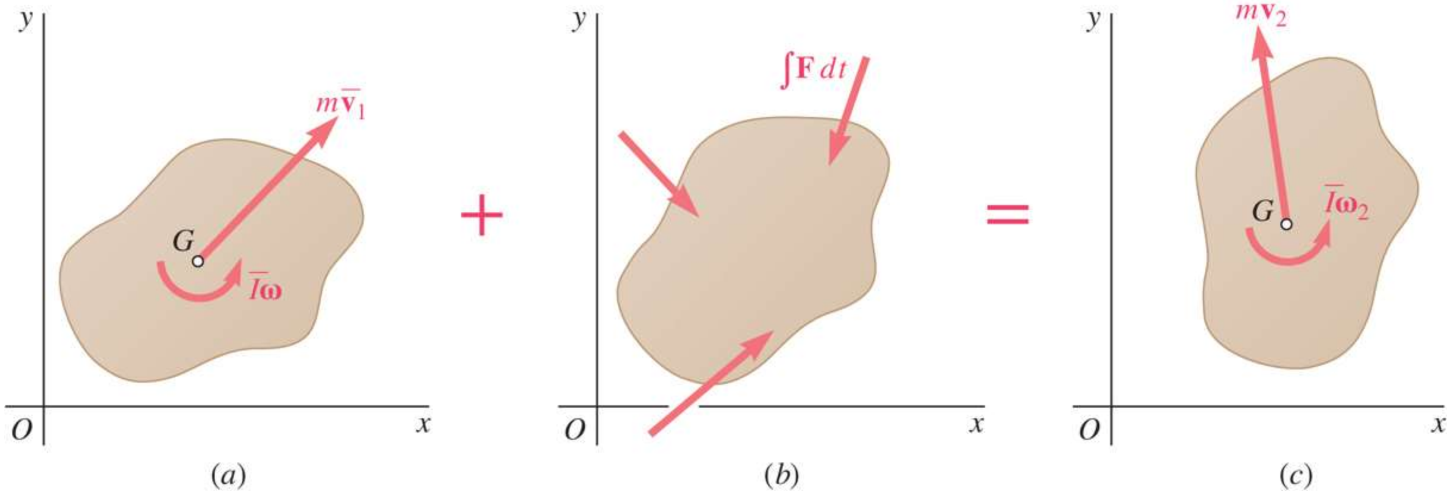
- For the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane,

$$\vec{H}_G = \bar{I}\omega$$



Principle of Impulse and Momentum

For plane motion problems, draw out an *impulse-momentum diagram*, (similar to a free-body diagram).

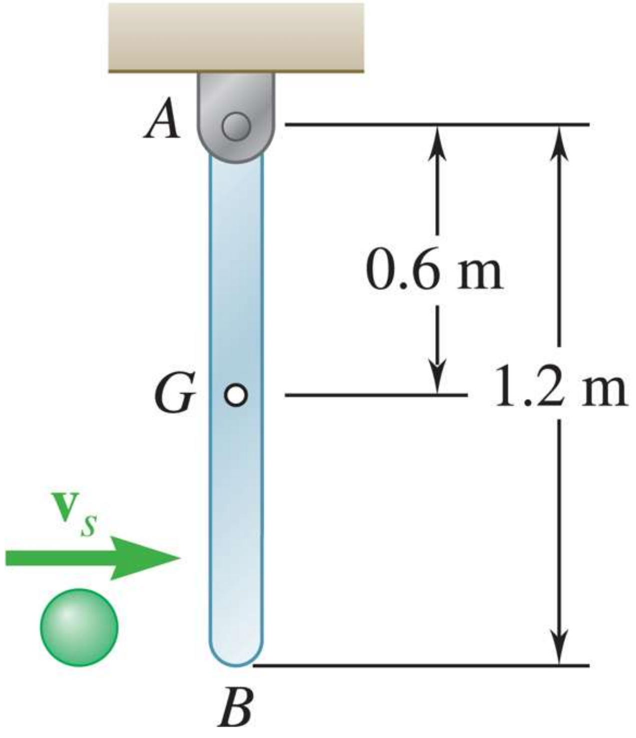


This leads to three equations of motion:

- summing and equating momenta and impulses in the *x* and *y* directions.
- summing and equating the moments of the momenta and impulses with respect to any given point (often choose *G*).

Impulse Momentum Diagrams 1

A sphere S hits a stationary bar AB and sticks to it. Draw the *impulse-momentum diagram* for the ball and bar separately; time 1 is immediately before the impact and time 2 is immediately after the impact.



Impulse Momentum Diagrams 2

Momentum of the ball before impact

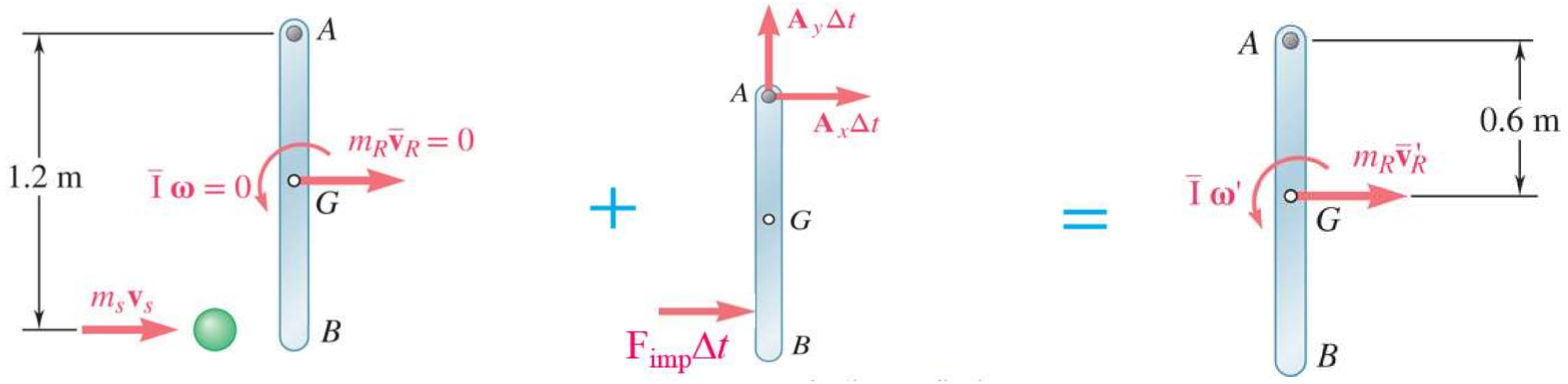
Impulse on ball

Momentum of the ball after impact

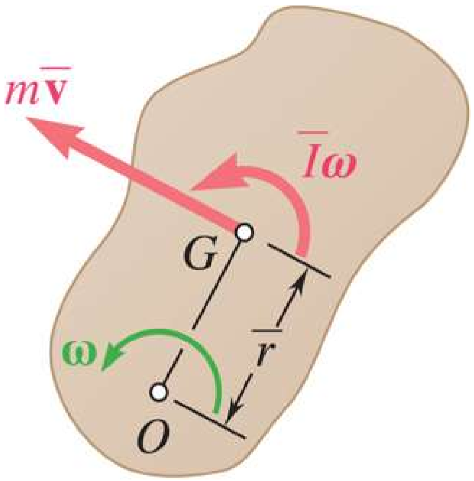
Momentum of the bar before impact

Impulse on bar

Momentum of the bar after impact



Principle of Impulse and Momentum



Fixed axis rotation:

- The angular momentum about O

$$\begin{aligned} I_O\omega &= \bar{I}\omega + (m\bar{v})\bar{r} \\ &= \bar{I}\omega + (m\bar{r}\omega)\bar{r} \\ &= \left(\bar{I} + m\bar{r}^2\right)\omega \end{aligned}$$

- Equating the moments of the momenta and impulses about O,

$$I_O\omega_1 + \sum \int_{t_1}^{t_2} M_O dt = I_O\omega_2$$

The pin forces at point O now contribute no moment to the equation

Systems of Rigid Bodies

- Motion of several rigid bodies can be analyzed by applying the principle of impulse and momentum to each body separately.

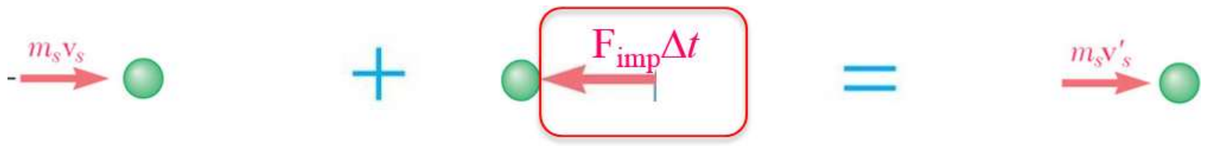
- For problems involving no more than three unknowns, it may be convenient to apply the principle of impulse and momentum to the system as a whole.

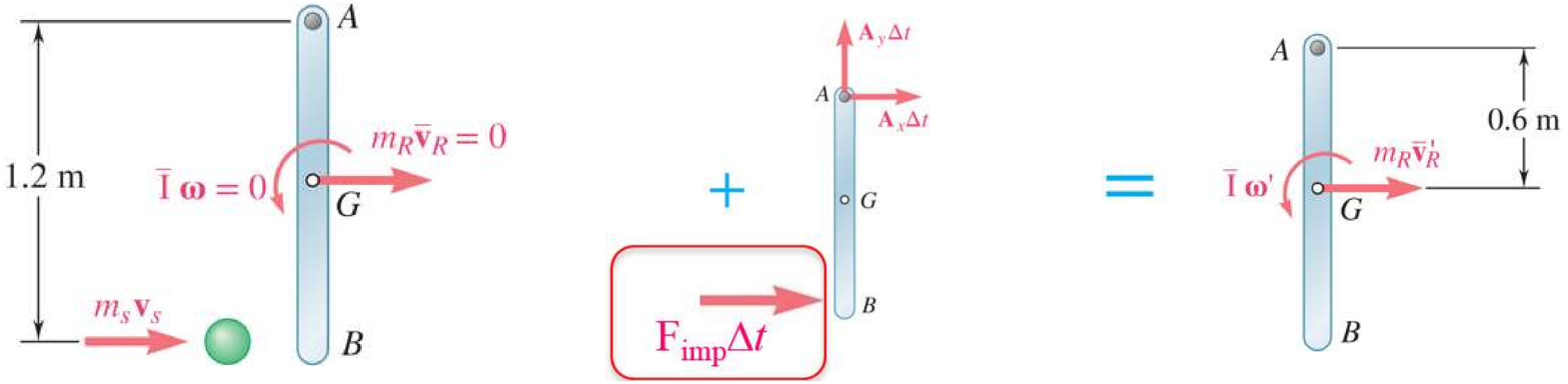
- For each moving part of the system, the diagrams of momenta should include a momentum vector and/or a momentum couple.

- Internal forces occur in equal and opposite pairs of vectors and generate impulses that cancel out.

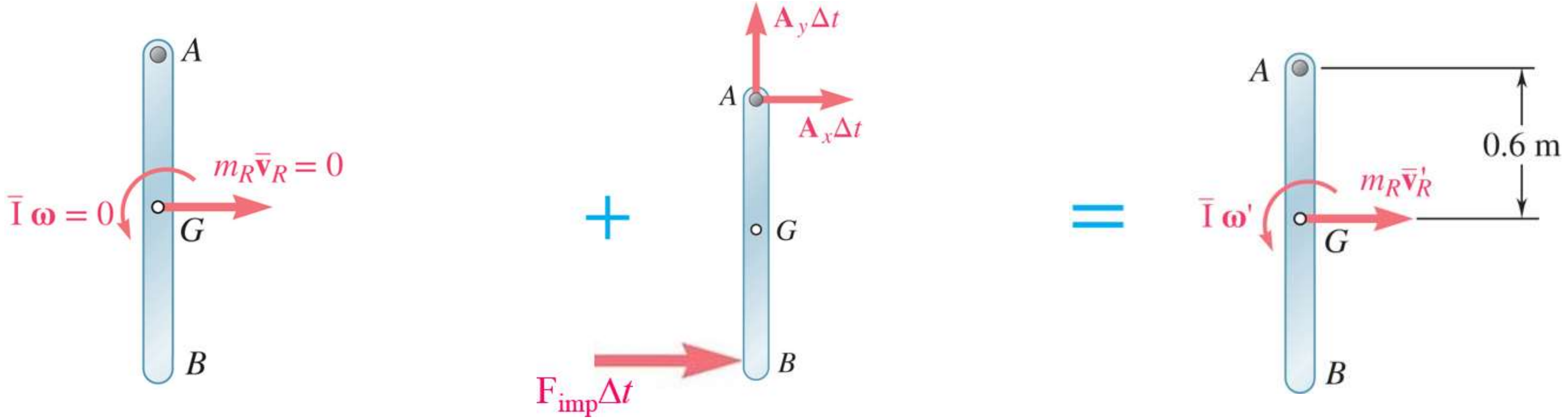
Practice

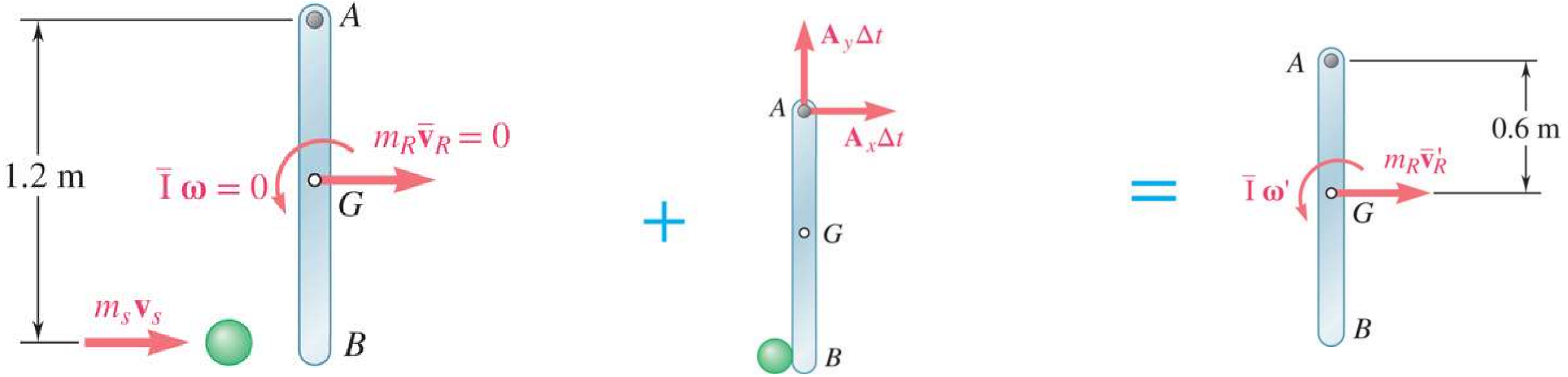
From the previous problem, notice that the impulse acting on the sphere is equal and opposite to the impulse acting on the bar. We can take advantage of this by drawing the *impulse-momentum* diagram of the entire system, as shown on the next slide.





Practice – Diagram for combined system





Conservation of Angular Momentum 1

The moments acting through the skater's center of gravity are negligible, so her angular momentum remains constant. She can adjust her spin rate by changing her moment of inertia.

$$I_G\omega_1 = I_G\omega_2$$

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Conservation of Angular Momentum

- When no external force acts on a rigid body or a system of rigid bodies, the system of momenta at t_1 is equipollent to the system at t_2. The total linear momentum and angular momentum about any point are conserved,

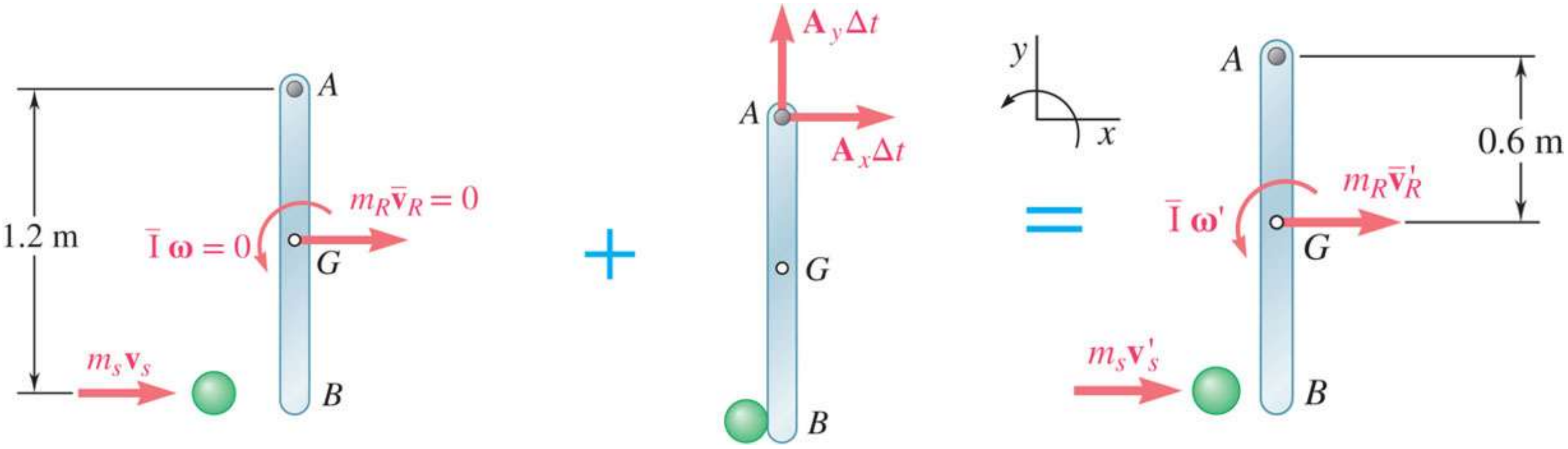
$$\vec{L}_1 = \vec{L}_2 \qquad (H_0)_1 = (H_0)_2$$

- When the sum of the angular impulses pass through *O,* the linear momentum may not be conserved, yet the angular momentum about *O* is conserved,

$$(H_0)_1 = (H_0)_2$$

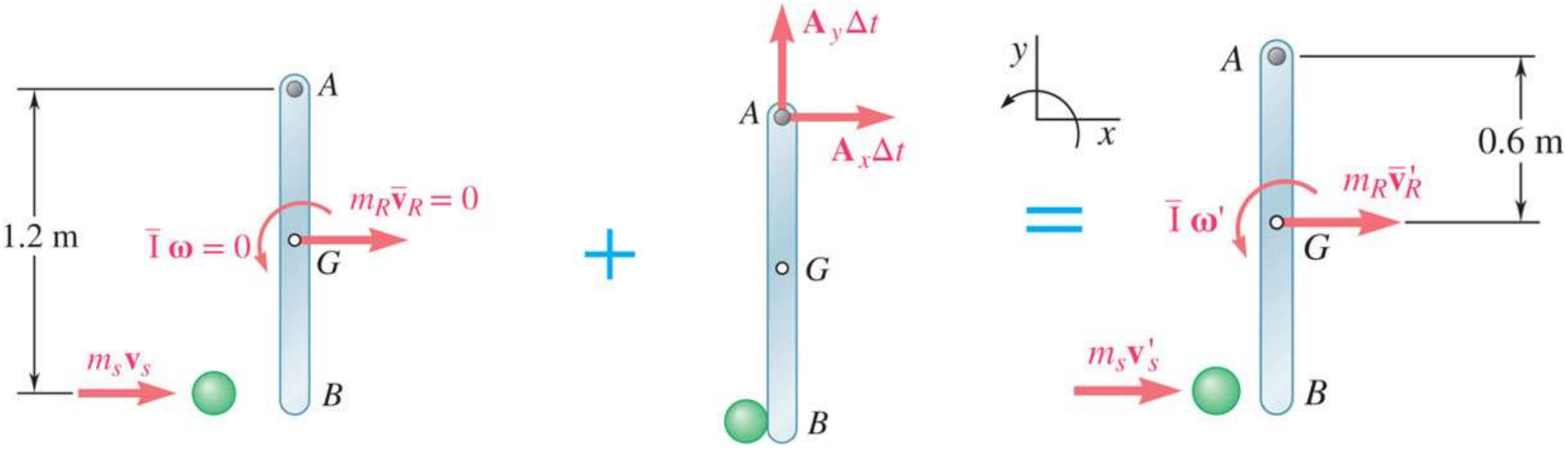
- Two additional equations may be written by summing x and y components of momenta and may be used to determine two unknown linear impulses, such as the impulses of the reaction components at a fixed point.

Concept Question



For the problem we looked at previously, is the angular momentum about G conserved?	Yes	No
For the problem we looked at previously, is the angular momentum about point A conserved?	Yes	No
For the problem we looked at previously, is the linear momentum of the system conserved?	Yes	No

Concept Question

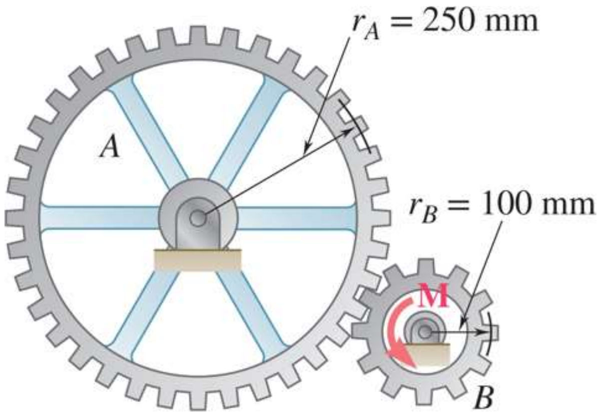


For the problem we looked at previously, is the angular momentum about G conserved? Yes [✓ No]

For the problem we looked at previously, is the angular momentum about point A conserved? [✓ Yes] No

For the problem we looked at previously, is the linear momentum of the system conserved? Yes [✓ No]

Sample Problem 17.7



$$m_A = 10\,\text{kg} \quad \bar{k}_A = 200\,\text{mm}$$
$$m_B = 3\,\text{kg} \quad \bar{k}_B = 80\,\text{mm}$$

The system is at rest when a moment of M = 6N ·m is applied to gear *B*.

Neglecting friction, *a*) determine the time required for gear *B* to reach an angular velocity of 600 rpm, and *b*) the tangential force exerted by gear *B* on gear *A*.

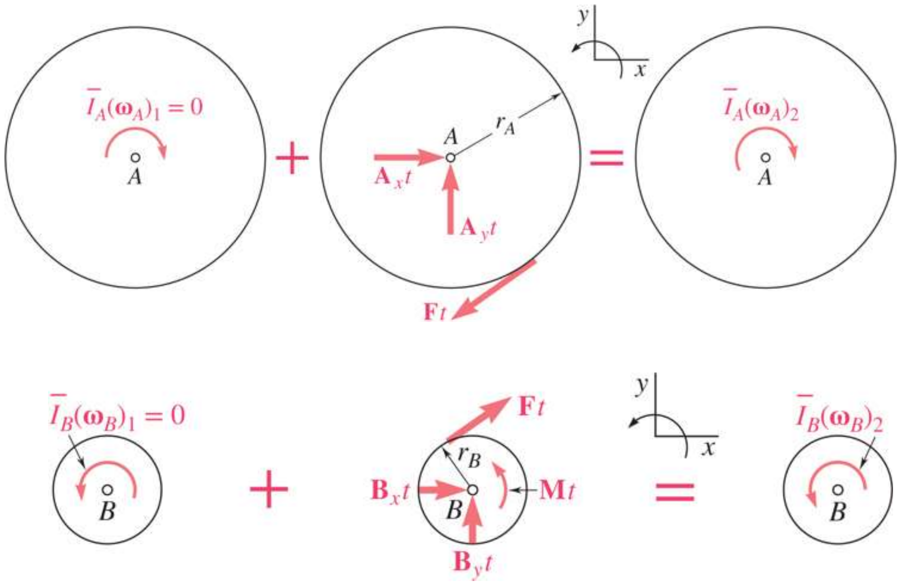
Strategy:

- Considering each gear separately, apply the method of impulse and momentum.
- Solve the angular momentum equations for the two gears simultaneously for the unknown time and tangential force.

Sample Problem 17.7

Modeling and Analysis:

- Considering each gear separately, apply the method of impulse and momentum.



+↺ moments about *A:*

$$0 - Ftr_A = -I_A(\omega_A)_2$$

$$Ft(0.250\,\text{m}) = (0.400\,\text{kg}\cdot\text{m})(25.1\,\text{rad/s})$$

$$Ft = 40.2\,\text{N}\cdot\text{s}$$

+↺ moments about *B:*

$$0 + Mt - Ftr_B = \bar{I}_B(\omega_B)_2$$

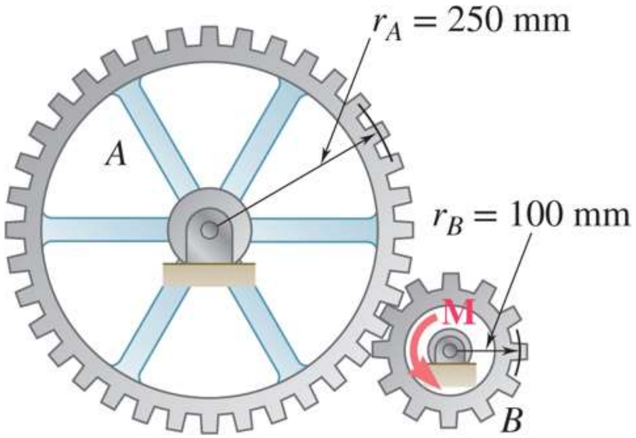
$$(6\,\text{N}\cdot\text{m})t - Ft(0.100\,\text{m})$$

$$= (0.0192\,\text{kg}\cdot\text{m}^2)(62.8\,\text{rad/s})$$

- Solve the angular momentum equations for the two gears simultaneously for the unknown time and tangential force.

$$\boxed{t = 0.871\,\text{s} \qquad F = 46.2\,\text{N}}$$

Sample Problem 17.7



$m_A = 10\,\text{kg} \quad \bar{k}_A = 200\,\text{mm}$

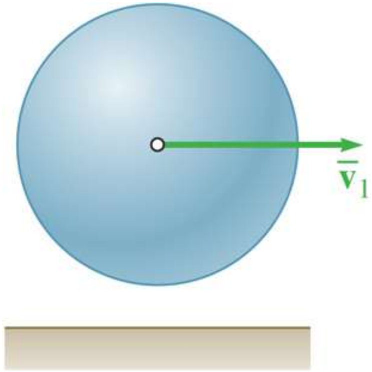
$m_B = 3\,\text{kg} \quad \bar{k}_B = 80\,\text{mm}$

Reflect and Think:

- This is the same answer obtained in Sample Prob. 17.2 by the method of work and energy, as you would expect. The difference is that in Sample Prob. 17.2, you were asked to find the number of revolutions, and in this problem, you were asked to find the time.

- What you are asked to find will often determine the best approach to use when solving a problem.

Sample Problem 17.8 1



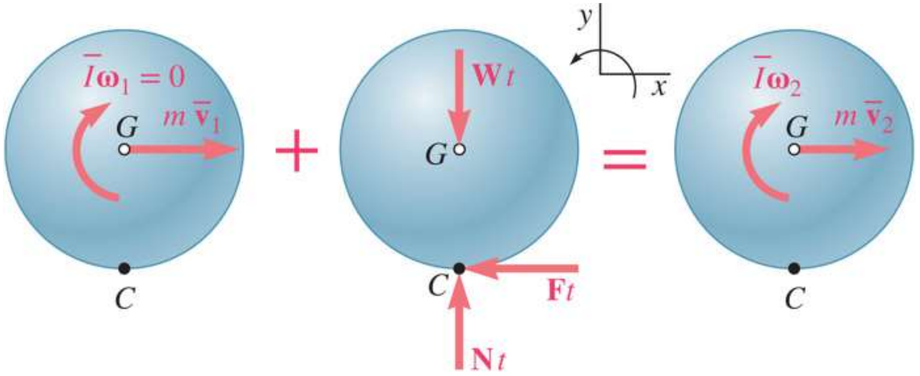
Uniform sphere of mass *m* and radius *r* is projected along a rough horizontal surface with a linear velocity $\bar{v}_1$ and no angular velocity. The coefficient of kinetic friction is μ_k.

Determine *a*) the time t_2 at which the sphere will start rolling without sliding and *b*) the linear and angular velocities of the sphere at time t_2.

Strategy:

- Apply principle of impulse and momentum to find variation of linear and angular velocities with time.
- Relate the linear and angular velocities when the sphere stops sliding by noting that the velocity of the point of contact is zero at that instant.
- Substitute for the linear and angular velocities and solve for the time at which sliding stops.
- Evaluate the linear and angular velocities at that instant.

Sample Problem 17.8 $_2$



$$Sys\ Momenta_1 + Sys\ Ext\ Imp_{1\text{-}2} = Sys\ Momenta_2$$

$+\uparrow$ y components:

$$Nt - Wt = 0$$

$$N = W = mg$$

$\xrightarrow{+}$ x components:

$$m\bar{v}_1 - Ft = m\bar{v}_2$$

$$m\bar{v}_1 - \mu_k mgt = m\bar{v}_2$$

$$\bar{v}_2 = \bar{v}_1 - \mu_k gt$$

$+\circlearrowleft$ moments about G:

$$Ft r = \bar{I}\omega_2$$

$$(\mu_k mg)tr = \left(\tfrac{2}{5}mr^2\right)\omega_2$$

$$\omega_2 = \frac{5}{2}\frac{\mu_k g}{r}t$$

Modeling and Analysis:

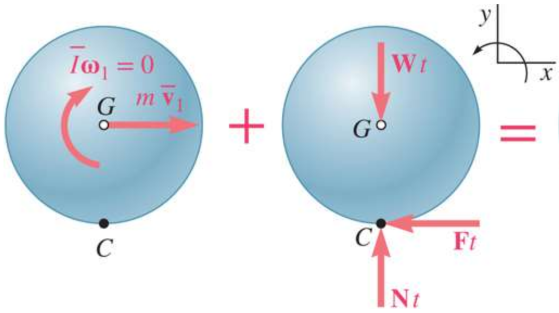
- Apply principle of impulse and momentum to find variation of linear and angular velocities with time.
- Relate linear and angular velocities when sphere stops sliding by noting that velocity of point of contact is zero at that instant.
- Substitute for the linear and angular velocities and solve for the time at which sliding stops.

$$\bar{v}_2 = r\omega_2$$

$$\bar{v}_1 - \mu_k gt = r\left(\frac{5}{2}\frac{\mu_k g}{r}t\right)$$

$$\boxed{t = \frac{2}{7}\frac{\bar{v}_1}{\mu_k g}}$$

Sample Problem 17.8 3



Sys Momenta$_1$ + *Sys Ext Imp*$_{1 \text{ to } 2}$ = *Sys Momenta*$_2$

$+\uparrow$ y components: $N = W = mg$

$\xrightarrow{+}$ x components: $\bar{v}_2 = \bar{v}_1 - \mu_k g t$

$+\circlearrowright$ moments about G: $\omega_2 = \dfrac{5}{2}\dfrac{\mu_k g}{r} t$

$$\bar{v}_2 = r\omega_2$$

$$\bar{v}_1 - \mu_k g t = r\left(\frac{5}{2}\frac{\mu_k g}{r} t\right) \qquad \boxed{t = \frac{2}{7}\frac{\bar{v}_1}{\mu_k g}}$$

- Evaluate the linear and angular velocities at that instant.

$$\bar{v}_2 = \bar{v}_1 - \mu_k g\left(\frac{2}{7}\frac{\bar{v}_1}{\mu_k g}\right)$$

$$\boxed{\bar{v}_2 = \frac{5}{7}\bar{v}_1 \rightarrow}$$

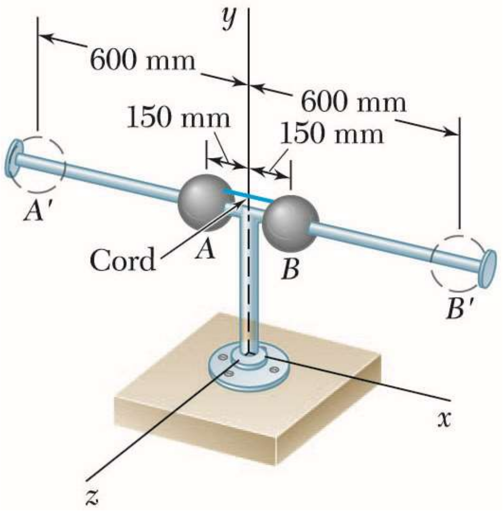
$$\omega_2 = \frac{5}{2}\frac{\mu_k g}{r}\left(\frac{2}{7}\frac{\bar{v}_1}{\mu_k g}\right)$$

$$\boxed{\omega_2 = \frac{5}{7}\frac{\bar{v}_1}{r} \circlearrowright}$$

Reflect and Think:

- This is the same answer obtained in Sample Prob. 16.6 by first dealing directly with force and acceleration and then applying kinematic relationships.

Sample Problem 17.9 1

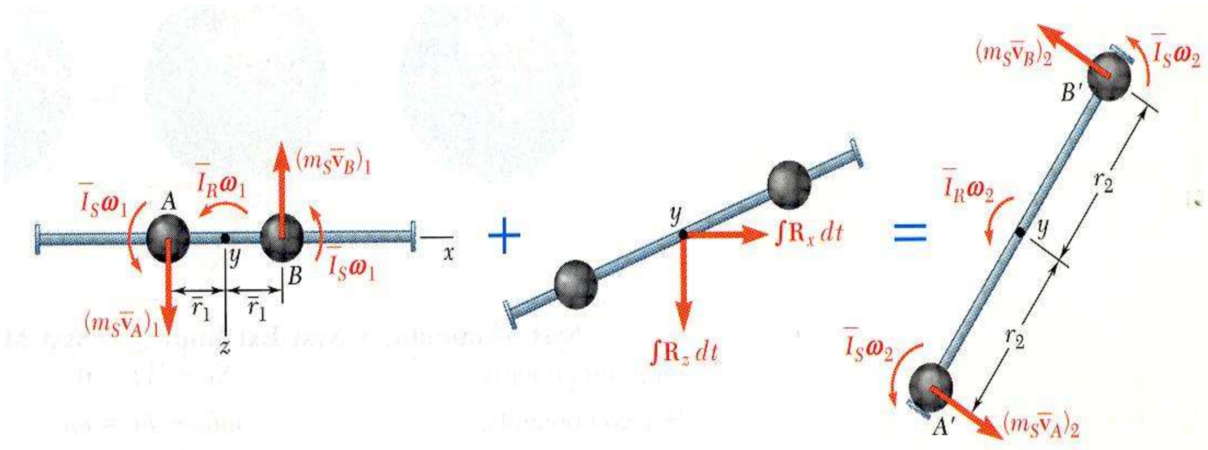


Two solid spheres (radius = 100 mm, *m* = 1 kg) are mounted on a spinning horizontal rod ($\bar{I}_R = 0.4\ \text{kg} \cdot \text{m}^2$, ω = 6 rad/sec) as shown. The balls are held together by a string which is suddenly cut. Determine *a*) angular velocity of the rod after the balls have moved to *A'* and *B'*, and *b*) the energy lost due to the plastic impact of the spheres and stops.

Strategy:

- Observing that none of the external forces produce a moment about the *y* axis, the angular momentum is conserved.
- Equate the initial and final angular momenta. Solve for the final angular velocity.
- The energy lost due to the plastic impact is equal to the change in kinetic energy of the system.

Sample Problem 17.9 2



$$Sys\ Momenta_1 + Sys\ Ext\ Imp_{1\text{-}2} = Sys\ Momenta_2$$

Modeling and Analysis:

- Observing that none of the external forces produce a moment about the *y* axis, the angular momentum is conserved.
- Equate the initial and final angular momenta. Solve for the final angular velocity.

$$2\left[(m_s\bar{r}_1\omega_1)\bar{r}_1 + \bar{I}_S\omega_1\right] + \bar{I}_R\omega_1 = 2\left[(m_s\bar{r}_2\omega_2)\bar{r}_2 + \bar{I}_S\omega_2\right] + \bar{I}_R\omega_2$$

$$\omega_2 = \omega_1 \frac{m_s\bar{r}_1^2 + \bar{I}_S + \bar{I}_R}{m_s\bar{r}_2^2 + \bar{I}_S + \bar{I}_R}$$

$$\omega_1 = 6\,\text{rad/s} \qquad \bar{I}_R = 0.4\ \text{kg}\cdot\text{m}^2$$

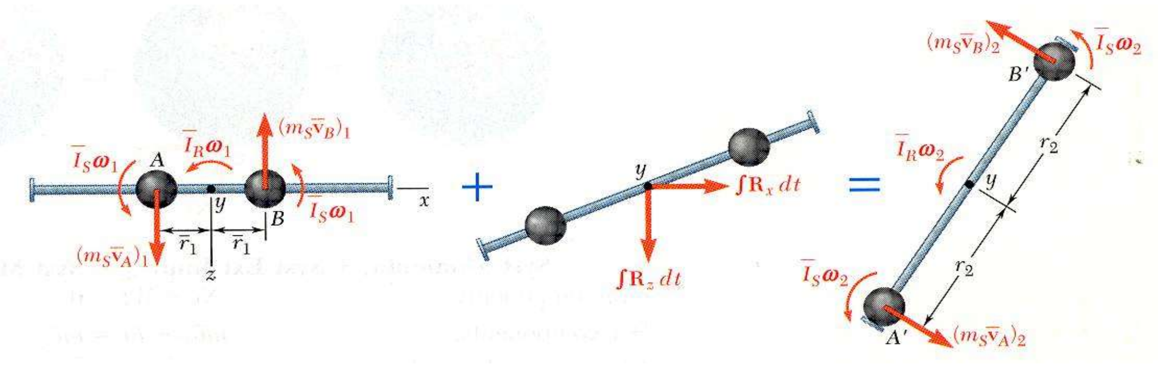
$$\bar{I}_S = \frac{2}{5}ma^2 = \frac{2}{5}(1\text{kg})(0.1\text{m})^2 = 0.04\text{kg}\cdot\text{m}^2$$

$$m_S\bar{r}_1^2 = (1\text{kg})(0.15\text{m})^2 = 0.0225\text{kg}\cdot\text{m}^2$$

$$m_S\bar{r}_2^2 = (1\text{kg})(0.6\text{m})^2 = 0.36\text{kg}\cdot\text{m}^2$$

$$\boxed{\omega_2 = 2.41\text{rad/s} \circlearrowleft}$$

Sample Problem 17.9 3



- The energy lost due to the plastic impact is equal to the change in kinetic energy of the system.

$\omega_1 = 6\,\text{rad/s}$ $\qquad$ $\omega_2 = 2.4096\text{rad/s}$

$\bar{I}_R = 0.4\ \text{kg}\cdot\text{m}^2$ $\qquad$ $\bar{I}_S = 0.004\text{kg}\cdot\text{m}^2$

$m_S\bar{r}_1^2 = 0.0225\text{kg}\cdot\text{m}^2$ $\qquad$ $m_S\bar{r}_2^2 = 0.36\text{kg}\cdot\text{m}^2$

$$T = 2\left(\tfrac{1}{2}m_S\bar{v}^2 + \tfrac{1}{2}\bar{I}_S\omega^2\right) + \tfrac{1}{2}\bar{I}_R\omega^2 = \tfrac{1}{2}\left(2m_S\bar{r}^2 + 2\bar{I}_S + \bar{I}_R\right)\omega^2$$

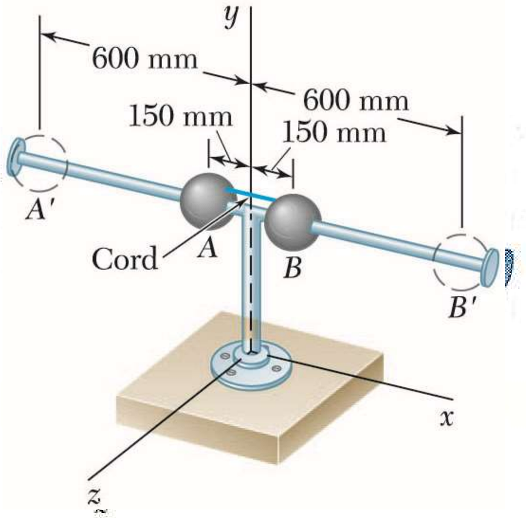
$$T_1 = \tfrac{1}{2}(0.453)(6)^2 = 8.154\,\text{J}$$

$$T_2 = \tfrac{1}{2}(1.128)(2.4096)^2 = 3.275\,\text{J}$$

$$\varDelta T = T_2 - T_1 = 8.154 - 3.275$$

$$\boxed{\Delta T = -4.88\ \text{J}}$$

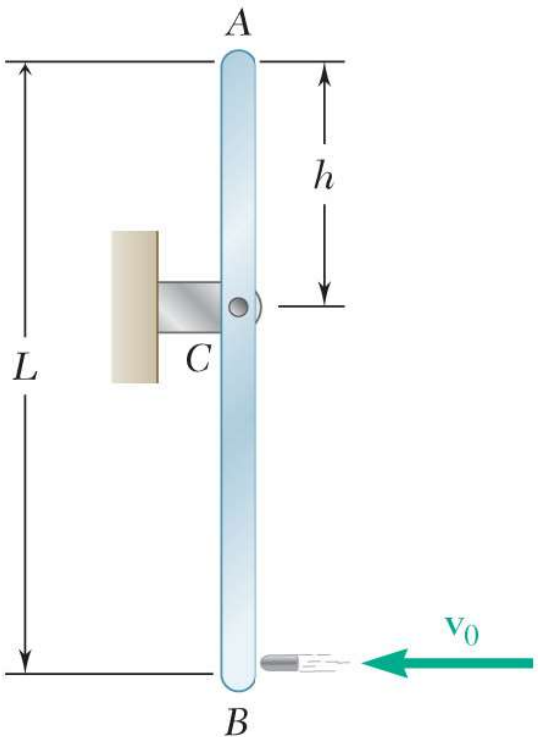
Sample Problem 17.9



Reflect and Think:

- As expected, when the spheres move outward, the angular velocity of the system decreases. This is similar to an ice skater who throws her arms outward to reduce her angular speed.

Group Problem Solving

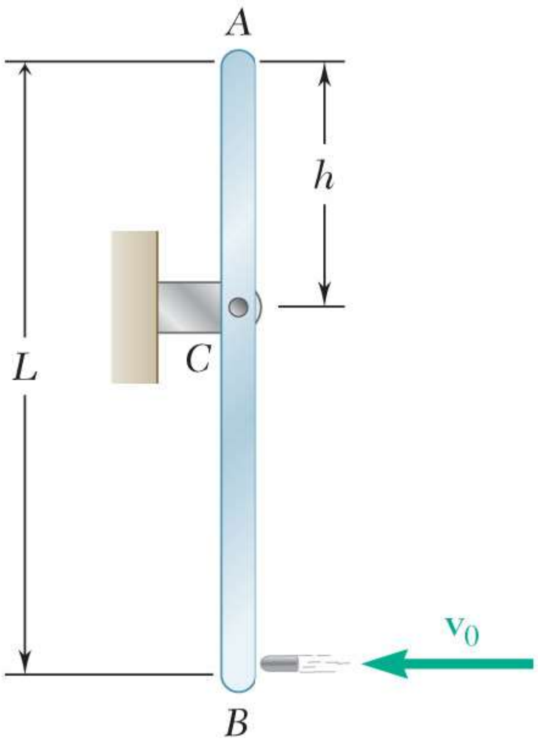


Strategy:

- Consider the projectile and bar as a single system. Apply the principle of impulse and momentum.
- The moments about *C* of the momenta and impulses provide a relation between the final angular velocity of the rod and velocity of the projectile.
- Use the principle of work-energy to determine the angle through which the bar swings.

A projectile weighing 40 g is fired with a horizontal velocity of 550 m/s into the lower end of a slender 7.5-kg bar of length *L*= 800 mm. Knowing that *h*= 300 mm and that the bar is initially at rest, determine the angular velocity of the bar when it reaches the horizontal position.

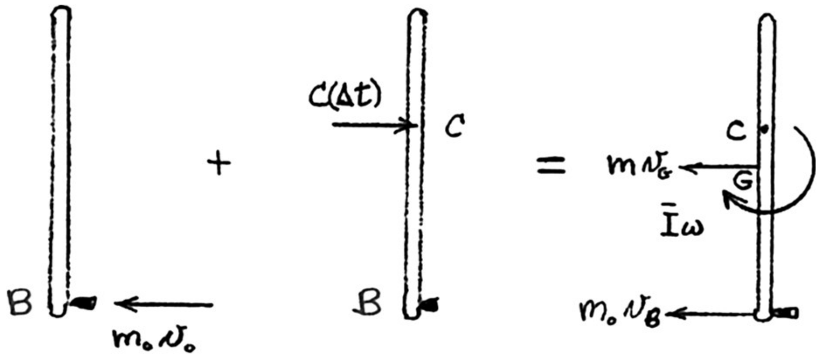
Group Problem Solving



Given: W_o= (0.04 kg)(9.81 m/s^2) = 0.3924 N, v_o= 550 m/s
W_{AB} = (7.5 kg)(9.81 m/s^2) = 73.575 N L= 0.8 m h= 0.3 m
Find: ω_{AB} when θ= 90°

Modeling and Analysis:

Draw the impulse momentum diagram



Apply the angular impulse momentum equation about point C

$$m_0 v_0 (L-h) = m_0 v_B (L-h) + I_C \omega$$

Or you could use the relationship:

$$m_0 v_0 (L-h) = m_0 v_B (L-h) + m v_0 \left(\frac{L}{2} - h\right) + \bar{I}\omega$$

Group Problem Solving

$$m_0 v_0 (L-h) = m_0 v_B (L-h) + I_C \omega$$

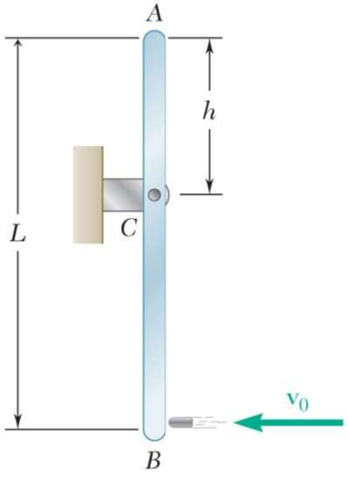
Relate v_B and ω (after the impact)

$$v_B = (L-h)\omega$$

Substitute into equation (1) and solve for ω

$$m_0 v_0 (L-h) = m_0 (L-h)^2 \omega + I_C \omega$$

$$\omega = \frac{m_0 v_0 (L-h)}{m_0 (L-h)^2 + I_C}$$



Find I_C

$$L = 0.8\,\text{m} \quad m = 7.5\,\text{kg}$$

$$I_C = \frac{1}{12} mL^2 + md^2 = \frac{1}{12}(7.5)(0.8)^2 + (7.5)(0.1)^2$$

$$I_C = 0.475\ \text{kg} \cdot \text{m}^2$$

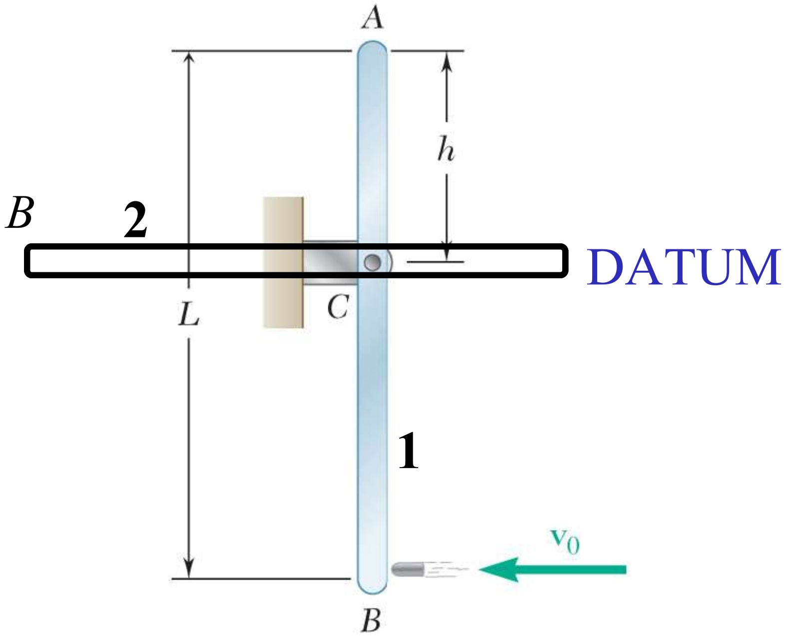
Substitute and solve

$$\omega = \frac{m_0 v_0 (L-h)}{m_0 (L-h)^2 + I_C} = \frac{(0.04\,\text{kg})(550\,\text{m/s})(0.8-0.3)}{(0.04\,\text{kg})(0.8-0.3)^2 + 0.475}$$

$$\omega_2 = 22.68\ \text{rad/s}$$

Group Problem Solving 8

Draw position 1 and 2, set your datum and apply the conservation of energy equation



$$T_2 + V_2 = T_3 + \cancel{V_3}$$

Find T_2

$$T_2 = \frac{1}{2} I_C \omega_2^2 = \frac{1}{2}(0.475)(22.68)^2$$

$$T_2 = 122.166 \text{ J}$$

Find V_2

$$V_2 = \text{m}_{AB}\text{g}\text{y}_{AB2} + \text{m}_O\text{g}\text{y}_{O2} = W_{AB}\text{y}_{AB2} + W_O\text{y}_{O2}$$

$$V_2 = -73.575(\tfrac{L}{2} - h) - 0.3924(L - h) = -73.575(0.1) - (0.3924)(0.5) = -7.5537 \text{ J}$$

Solve for ω_3

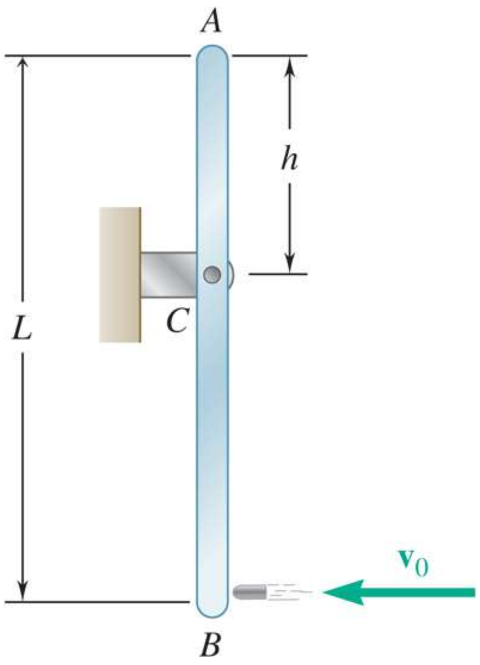
$$T_3 = \frac{1}{2} I_C \omega_3^2 = T_2 + V_2$$

$$\frac{1}{2}(0.475)\omega_3^2 = 122.166 - 7.5537$$

$$\boxed{\omega_3 = 22.0 \text{ rad/s}}$$

Concept Question

For the previous problem, how would you determine the reaction forces at C when the bar is horizontal?



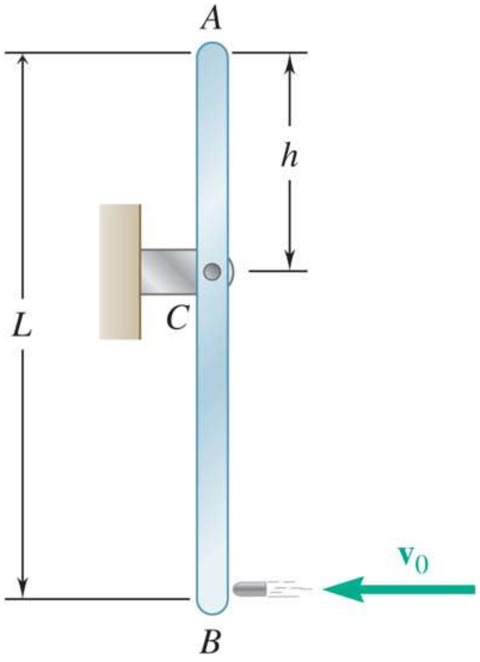
a) Apply linear-momentum to solve for $C_x\Delta t$ and $C_y\Delta t$.

b) Use work-energy to determine the work done by the moment at C.

c) Use sum of forces and sum of moments equations when the bar is horizontal.

Concept Question

For the previous problem, how would you determine the reaction forces at C when the bar is horizontal?



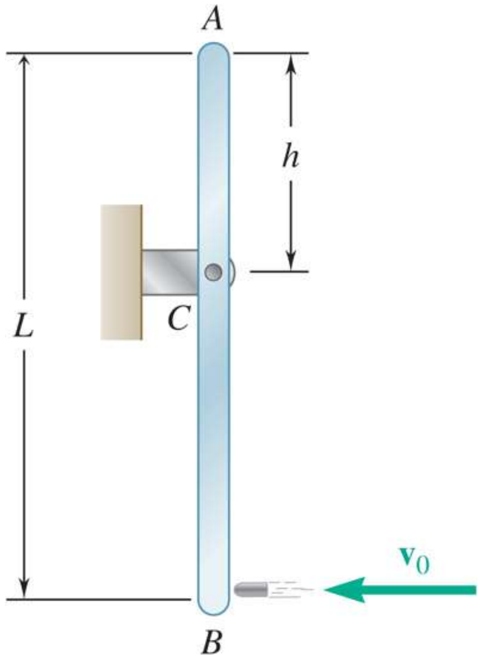
a) Apply linear-momentum to solve for $C_x\Delta t$ and $C_y\Delta t$.

b) Use work-energy to determine the work done by the moment at C.

c) Use sum of forces and sum of moments equations when the bar is horizontal.

Concept Question

For the previous problem, what would happen if the coefficient of restitution between the projectile and bar was 1.0 instead of zero?



a) The angular velocity after impact would be bigger.

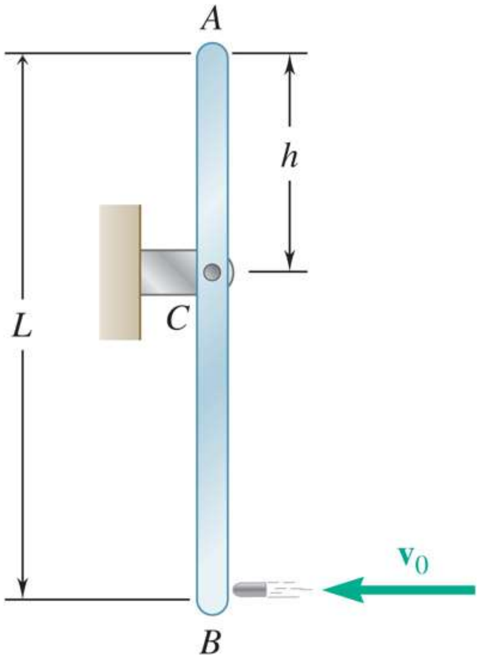
b) The angular velocity after impact would be smaller.

c) The angular velocity after impact would be the same.

c) Not enough information to tell.

Concept Question

For the previous problem, what would happen if the coefficient of restitution between the projectile and bar was 1.0 instead of zero?



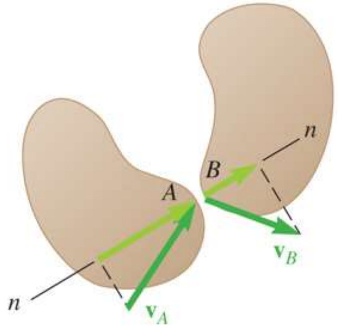
a) The angular velocity after impact would be bigger.

b) The angular velocity after impact would be smaller.

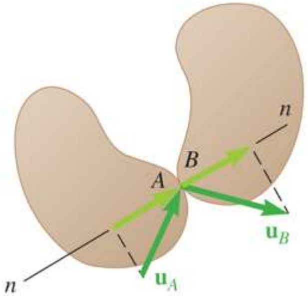
c) The angular velocity after impact would be the same.

d) Not enough information to tell.

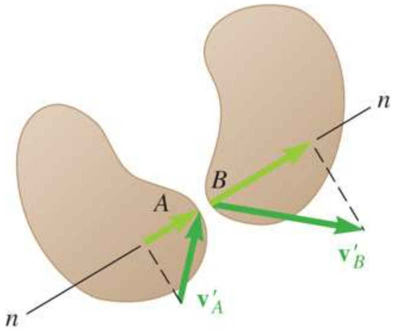
Eccentric Impact



(a)



(b)



(c)

$$(\vec{u}_A)_n = (\vec{u}_B)_n$$

Period of deformation

$$Impulse = \int \vec{R} dt$$

Period of restitution

$$Impulse = \int \vec{P} dt$$

- Principle of impulse and momentum is supplemented by

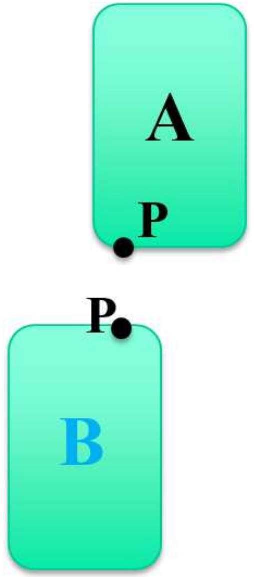
$$e = coefficient\ of\ restitution = \frac{\int \vec{R} dt}{\int \vec{P} dt}$$

$$= \frac{(v'_B)_n - (v'_A)_n}{(v_A)_n - (v_B)_n}$$

These velocities are for the points of impact

Concept Question

The cars collide, hitting at point P as shown. Which of the following can you use to help analyze the collision?



a) The linear momentum of car A is conserved.

b) The linear momentum of the combined two cars is conserved

c) The total kinetic energy before the impact equals the total kinetic energy after the impact

d) The angular momentum about the CG of car B is conserved

Concept Question

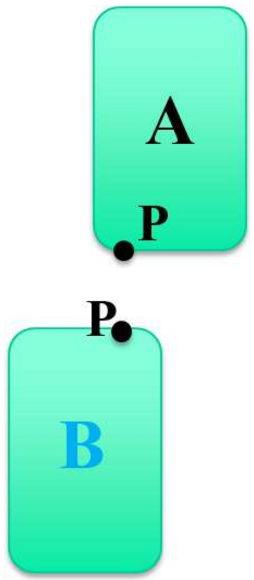
The cars collide, hitting at point P as shown. Which of the following can you use to help analyze the collision?

a) The linear momentum of car A is conserved.

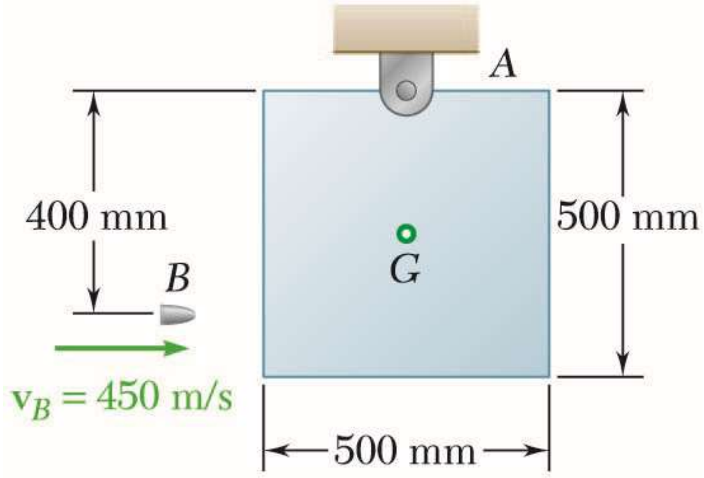
b) The linear momentum of the combined two cars is conserved

c) The total kinetic energy before the impact equals the total kinetic energy after the impact

d) The angular momentum about the CG of car B is conserved



Sample Problem 17.11



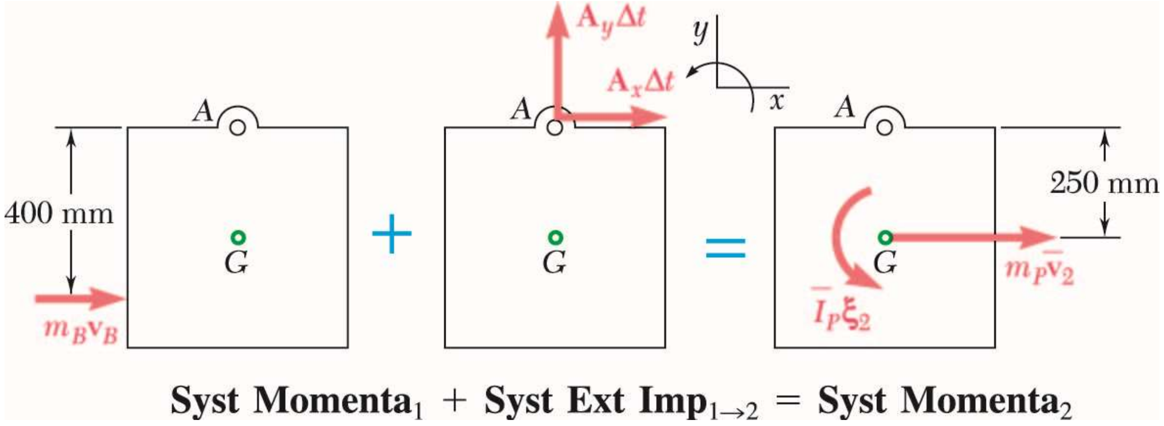
A 25-g bullet is fired into the side of a 10-kg square panel which is initially at rest.

Determine *a*) the angular velocity of the panel immediately after the bullet becomes embedded and *b*) the impulsive reaction at *A*, assuming that the bullet becomes embedded in 0.0006 s.

Strategy:

- Consider a system consisting of the bullet and panel. Apply the principle of impulse and momentum.
- The final angular velocity is found from the moments of the momenta and impulses about *A*.
- The reaction at *A* is found from the horizontal and vertical momenta and impulses.

Sample Problem 17.11



Modeling and Analysis:

- Consider a system consisting of the bullet and panel. Apply the principle of impulse and momentum.
- The final angular velocity is found from the moments of the momenta and impulses about A.

$+\circlearrowleft$ moments about A:

$$m_B v_B(0.4\text{m}) + 0 = m_p \bar{v}_2(0.25\text{m}) + \bar{I}_P\omega_2$$

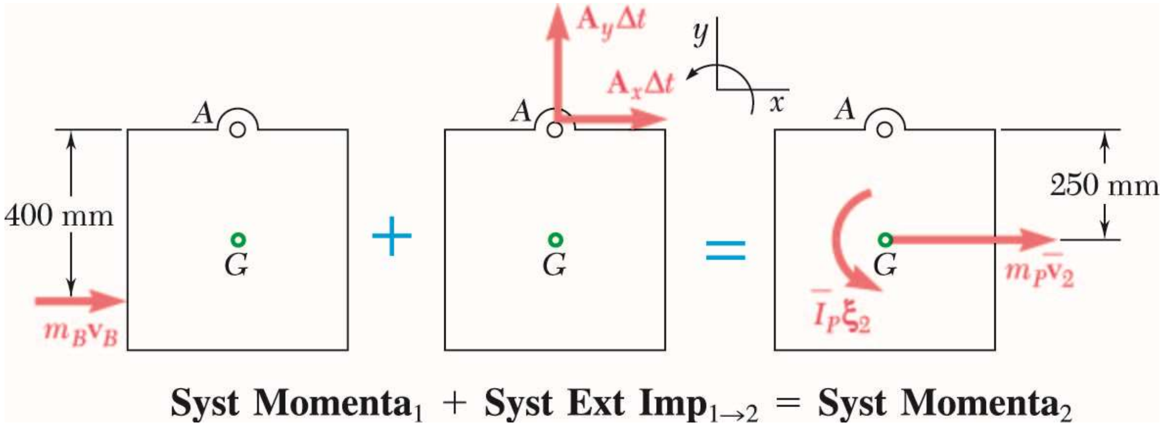
$$\bar{v}_2 = (0.25\text{m})\omega_2 \qquad \bar{I}_P = \frac{1}{6}m_P b^2 = \frac{1}{6}(10\text{kg})(0.5\text{m})^2 = 0.417\text{kg}\cdot\text{m}^2$$

$$(0.025)(450)(0.4) = (10)(0.25\omega_2)(0.25) + 0.417\omega_2$$

$$\omega_2 = 4.32\text{rad/s}$$
$$\bar{v}_2 = (0.25)\omega_2 = 1.08\text{m/s}$$

$$\boxed{\omega_2 = 4.32\text{rad/s} \circlearrowleft}$$

Sample Problem 17.11



- The reactions at *A* are found from the horizontal and vertical momenta and impulses.

$\omega_2 = 4.32\text{rad/s}$ $\quad\bar{v}_2 = (0.25)\omega_2 = 1.08\text{m/s}$

$\xrightarrow{+}$ *x* components:

$$m_B v_B + A_x \Delta t = m_p \bar{v}_2$$
$$(0.025)(450) + A_x(0.0006) = (10)(1.08)$$

$$A_x = -750\,\text{N} \qquad \boxed{A_x = 750\,\text{N} \leftarrow}$$

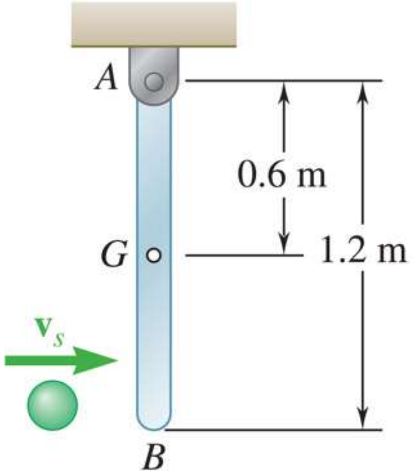
$+\uparrow$ *y* components:

$$0 + A_y \Delta t = 0 \qquad \boxed{A_y = 0}$$

Reflect and Think:

- The speed of the bullet is in the range of a modern high-performance rifle. Notice that the reaction at A is over 5000 times the weight of the bullet and over 10 times the weight of the plate.

Sample Problem 17.13



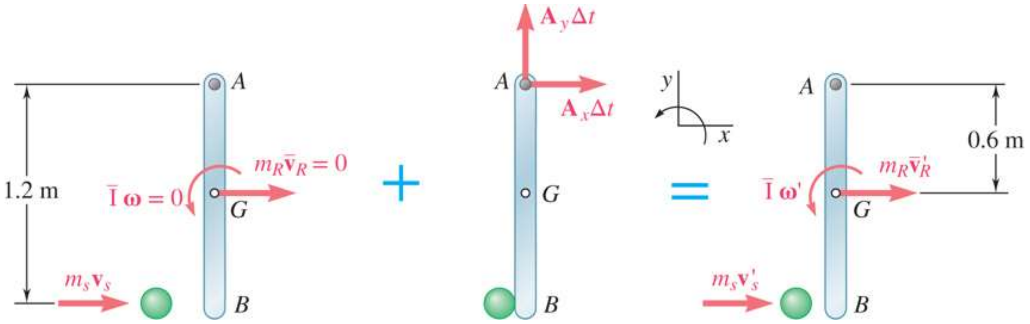
A 2-kg sphere with an initial velocity of 5 m/s strikes the lower end of an 8-kg rod *AB*. The rod is hinged at *A* and initially at rest. The coefficient of restitution between the rod and sphere is 0.8.

Determine the angular velocity of the rod and the velocity of the sphere immediately after impact.

Strategy:

- Consider the sphere and rod as a single system. Apply the principle of impulse and momentum.
- The moments about *A* of the momenta and impulses provide a relation between the final angular velocity of the rod and velocity of the sphere.
- The definition of the coefficient of restitution provides a second relationship between the final angular velocity of the rod and velocity of the sphere.
- Solve the two relations simultaneously for the angular velocity of the rod and velocity of the sphere.

Sample Problem 17.13 2



Modeling and Analysis:

- Consider the sphere and rod as a single system. Apply the principle of impulse and momentum.
- The moments about A of the momenta and impulses provide a relation between the final angular velocity of the rod and velocity of the rod.

$+\circlearrowleft$ moments about A:

$$m_s v_s (1.2\,\text{m}) = m_s v'_s (1.2\,\text{m}) + m_R \bar{v}'_R (0.6\,\text{m}) + \bar{I}\omega'$$

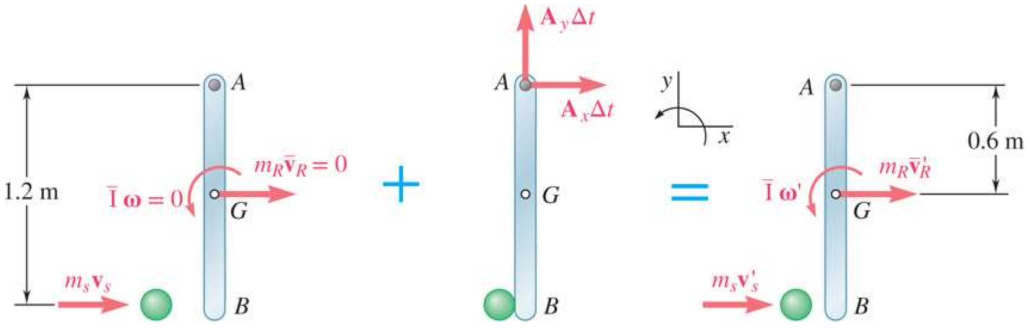
$$\bar{v}'_R = \bar{r}\omega' = (0.6\,\text{m})\omega'$$

$$\bar{I} = \tfrac{1}{12} mL^2 = \tfrac{1}{12}(8\,\text{kg})(1.2\,\text{m})^2 = 0.96\,\text{kg}\cdot\text{m}^2$$

$$(2\,\text{kg})(5\,\text{m/s})(1.2\,\text{m}) = (2\,\text{kg})v'_s(1.2\,\text{m}) + (8\,\text{kg})(0.6\,\text{m})\omega'(0.6\,\text{m}) + \left(0.96\,\text{kg}\cdot\text{m}^2\right)\omega'$$

$$12 = 2.4 v'_s + 3.84\omega'$$

Sample Problem 17.13



+↺ Moments about *A*:

$$12 = 2.4v'_s + 3.84\omega'$$

$\xrightarrow{+}$ Relative velocities:

$$v'_B - v'_s = e(v_B - v_s)$$

$$(1.2\,\mathrm{m})\omega' - v'_s = 0.8(5\,\mathrm{m/s})$$

Solving,

$$\omega' = 3.21\,\mathrm{rad/s} \qquad \boxed{\omega' = 3.21\,\mathrm{rad/s}\ ↺}$$

$$v'_s = -0.143\,\mathrm{m/s} \qquad \boxed{\mathrm{v}'_s = 0.143\,\mathrm{m/s} \leftarrow}$$

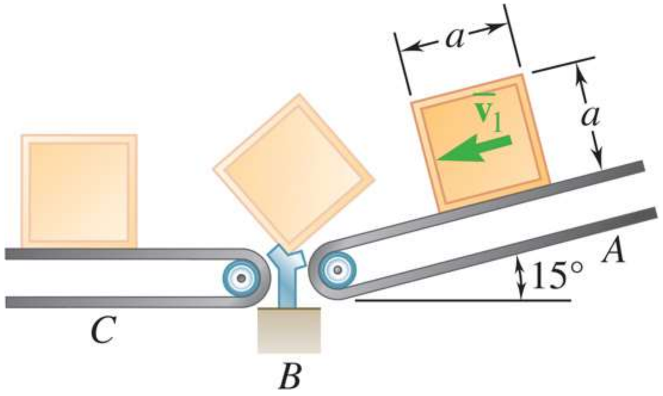
- The definition of the coefficient of restitution provides a second relationship between the final angular velocity of the rod and velocity of the sphere.

- Solve the two relations simultaneously for the angular velocity of the rod and velocity of the sphere.

Reflect and Think

- The negative value for the velocity of the sphere after impact means that it bounces back to the left. Given the masses of the sphere and the rod, this seems reasonable

Sample Problem 17.14



A square package of mass m moves down conveyor belt A with constant velocity. At the end of the conveyor, the corner of the package strikes a rigid support at B. The impact is perfectly plastic.

Derive an expression for the minimum velocity of conveyor belt A for which the package will rotate about B and reach conveyor belt C.

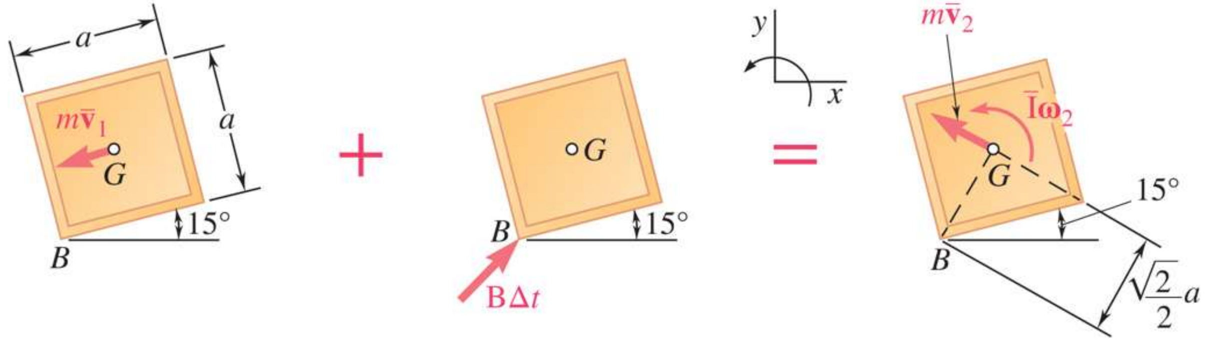
Strategy:

- Apply the principle of impulse and momentum to relate the velocity of the package on conveyor belt A before the impact at B to the angular velocity about B after impact.
- Apply the principle of conservation of energy to determine the minimum initial angular velocity such that the mass center of the package will reach a position directly above B.
- Relate the required angular velocity to the velocity of conveyor belt A.

Sample Problem 17.14 2

Modeling and Analysis:

- Apply the principle of impulse and momentum to relate the velocity of the package on conveyor belt *A* before the impact at *B* to angular velocity about *B* after impact.



+↻ Moments about *B*:

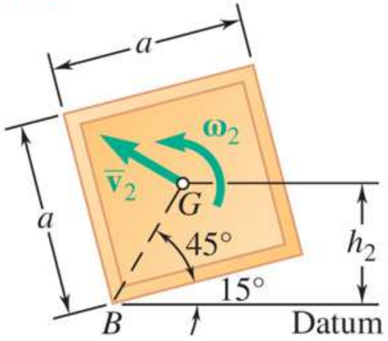
$$(m\bar{v}_1)\left(\tfrac{1}{2}a\right)+0=(m\bar{v}_2)\left(\tfrac{\sqrt{2}}{2}a\right)+\bar{I}\omega_2 \qquad \bar{v}_2=\left(\tfrac{\sqrt{2}}{2}a\right)\omega_2 \qquad \bar{I}=\tfrac{1}{6}ma^2$$

$$(m\bar{v}_1)\left(\tfrac{1}{2}a\right)+0=m\left(\tfrac{\sqrt{2}}{2}a\omega_2\right)\left(\tfrac{\sqrt{2}}{2}a\right)+\left(\tfrac{1}{6}ma^2\right)\omega_2$$

$$\bar{v}_1=\tfrac{4}{3}a\omega_2$$

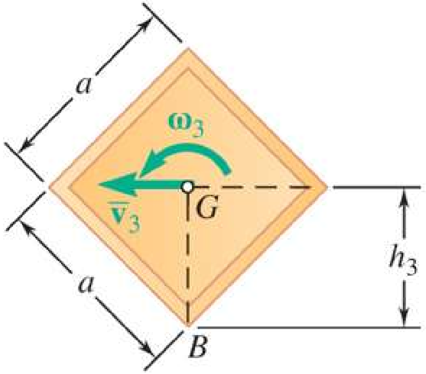
Sample Problem 17.14 3

Position 2



$$h_2 = (GB)\sin(45° + 15°)$$
$$= \left(\frac{\sqrt{2}}{2}a\right)\sin 60° = 0.612a$$

Position 3



$$h_3 = \frac{\sqrt{2}}{2}a = 0.707a$$

- Apply the principle of conservation of energy to determine the minimum initial angular velocity such that the mass center of the package will reach a position directly above *B*.

$$T_2 + V_2 = T_3 + V_3$$

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2$$

$$= \tfrac{1}{2}m\left(\tfrac{\sqrt{2}}{2}a\omega_2\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{6}ma^2\right)\omega_2^2 = \tfrac{1}{3}ma^2\omega_2^2$$

$$V_2 = Wh_2$$

$T_3 = 0$ (solving for the minimum w_2)

$$V_3 = Wh_3$$

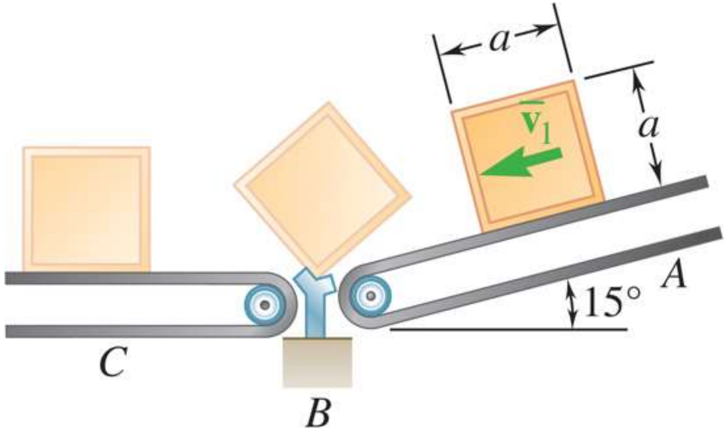
$$\tfrac{1}{3}ma^2\omega_2^2 + Wh_2 = 0 + Wh_3$$

$$\omega_2^2 = \frac{3W}{ma^2}(h_3 - h_2) = \frac{3g}{a^2}(0.707a - 0.612a) = \sqrt{0.285\,g/a}$$

$$\bar{v}_1 = \tfrac{4}{3}a\omega_2 = \tfrac{4}{3}a\sqrt{0.285\,g/a}$$

$$\boxed{\bar{v}_1 = 0.712\sqrt{ga}}$$

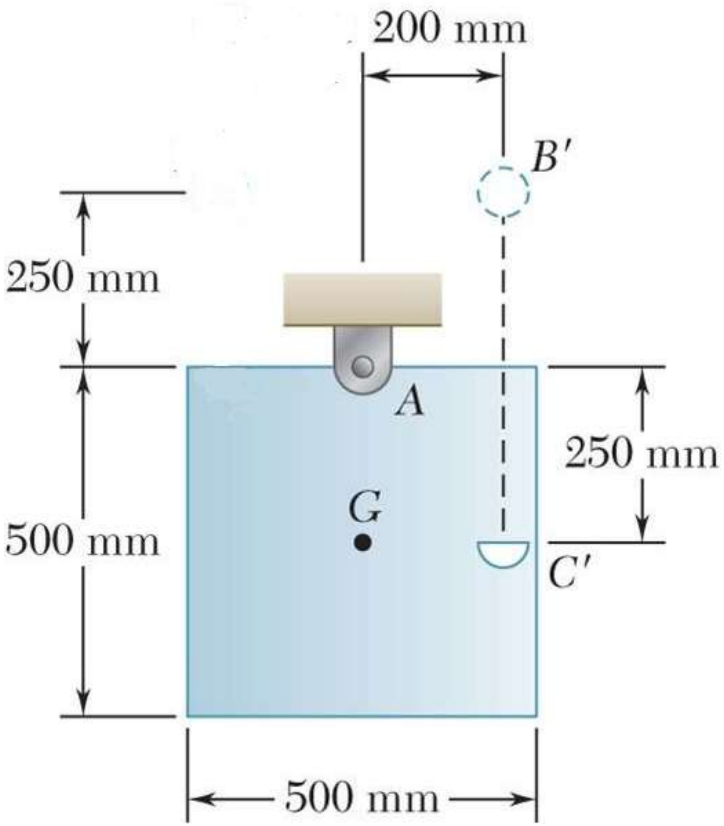
Sample Problem 17.14



Reflect and Think:

- The combination of energy and momentum methods is typical of many design analyses. If you had been interested in determining the reaction at B immediately after the impact or at some other point in the motion, you would have needed to draw a free-body diagram and kinetic diagram and apply Newton’s second law.

Group Problem Solving

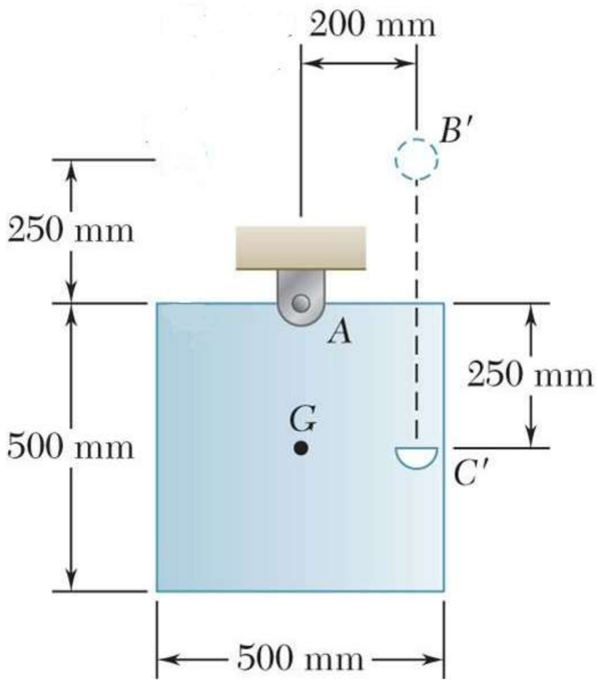


Strategy:

- Consider the sphere and panel as a single system. Apply the principle of impulse and momentum.
- The moments about *A* of the momenta and impulses provide a relation between the angular velocity of the panel and velocity of the sphere.
- Use the principle of work-energy to determine the angle through which the panel swings.

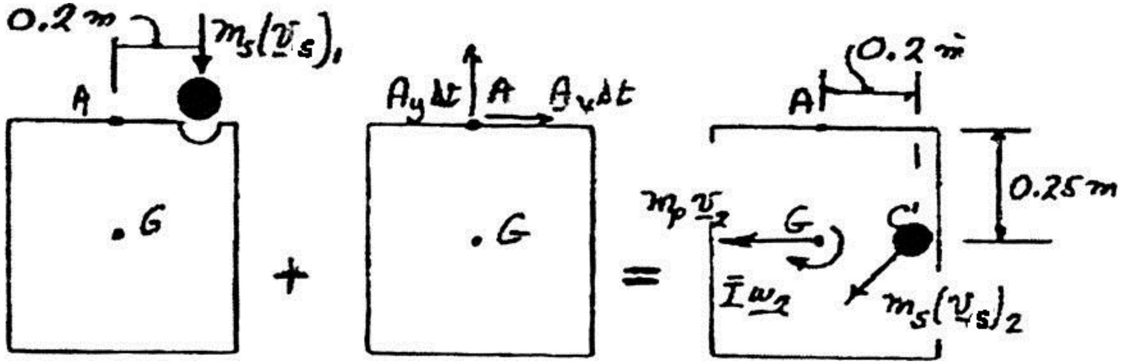
An 8-kg wooden panel *P* is suspended from a pin support at *A* and is initially at rest. A 2-kg metal sphere *S* is released from rest at *B*′ and falls into a hemispherical cup *C*′ attached to the panel at the same level as the mass center *G*. Assuming that the impact is perfectly plastic, determine the angular velocity of the panel immediately after the impact.

Group Problem Solving



Modeling and Analysis:

Draw the impulse momentum diagram



Apply the angular impulse momentum equation about point A

Given: m_S= 2 kg, m_P = 8 kg, h_S= 0.250 m, e= 0.

Find: Angle θ through which the panel and sphere swing after the impact

$$m_S(v_{C'})_1(0.2\text{ m})+0=m_S(v_{C'})_2(AC')+\underline{\bar{I}\omega_2+m_P\bar{v}_2(0.25\text{ m})}$$

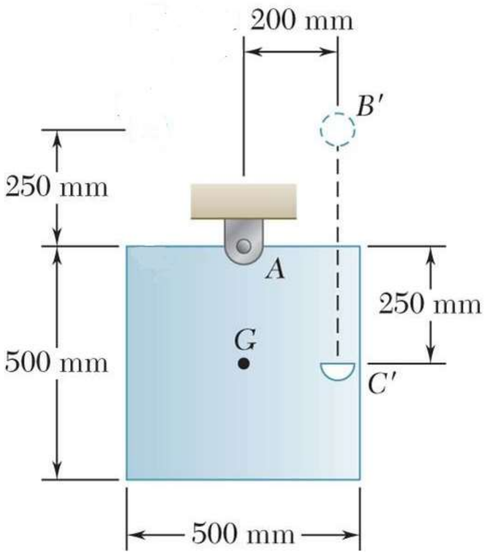
↑ H_A of sphere before impact

↑ H_A of sphere after impact

↑ H_A of panel after impact

Group Problem Solving

$$m_S(v_{C'})_1(0.2\text{ m}) + 0 = m_S(v_{C'})_2(AC') + \bar{I}\omega_2 + m_P\bar{v}_2(0.25\text{ m})$$



Determine velocity of sphere at impact $(v_S)_1$

You can apply work-energy or kinematics

$$(v_S)_1 = \sqrt{2gy}$$
$$= \sqrt{2(9.81\text{ m/s}^2)(0.5\text{ m})}$$
$$= 3.1321\text{ m/s}$$

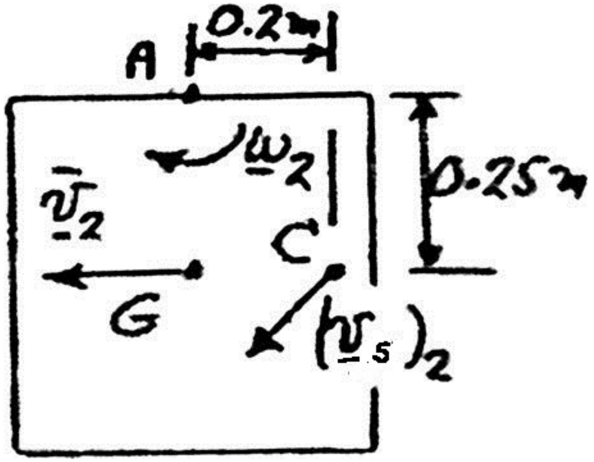
Determine velocity of sphere after impact in terms of ω_2

$$(\mathbf{v}_S)_2 = AC'\omega_2$$

$$AC' = \sqrt{(0.2)^2 + (0.25)^2} = 0.32016\text{ m}$$

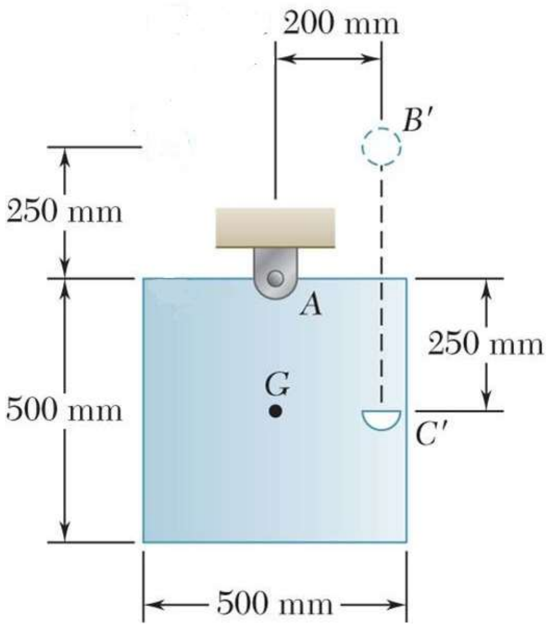
$$(\mathbf{v}_S)_2 = 0.32016\omega_2$$

(perpendicular to *AC*.)



Group Problem Solving

$$m_S(v_{C'})_1(0.2\text{ m})+0=m_S(v_{C'})_2(AC')+\bar{I}\omega_2+m_P\bar{v}_2(0.25\text{ m})$$



Determine mass moment of inertia for panel

$$\bar{I}=\frac{1}{6}m_P(0.5\text{ m})^2=\frac{1}{6}(8)(0.5)^2=0.3333\text{ kg}\cdot\text{m}^2$$

Substitute into H equation and solve for ω_2

$$m_S(v_{C'})_1(0.2\text{ m})+0=m_S(v_{C'})_2(AC')+\bar{I}\omega_2+m_P\bar{v}_2(0.25\text{ m})$$

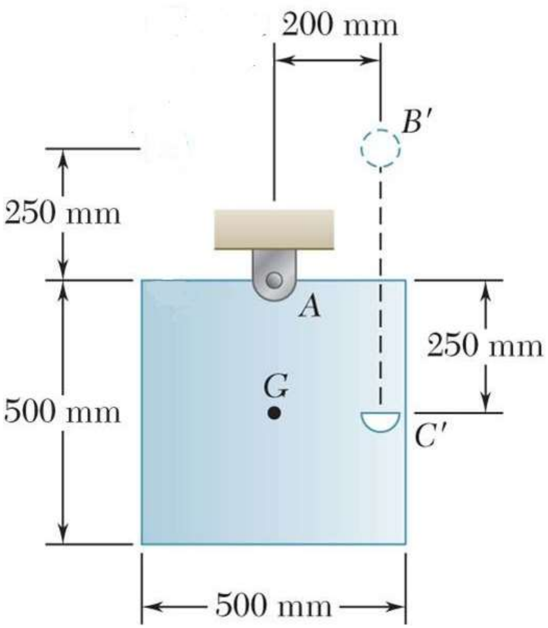
$$(2\text{ kg})(3.1321\text{ m/s})(0.2\text{ m})=(2\text{ kg})(0.32016\omega_2)(0.32016\text{ m})+0.3333\omega_2+(8\text{ kg})(0.25\text{ m})^2\,\omega_2$$

$$1.25284=(0.2050+0.3333+0.500)\omega_2$$

$$\boxed{\omega_2=1.207\text{ rad/s}}$$

Concept Question

For the previous problem, what would you do if you wanted to determine how high up the panel swung after the impact?



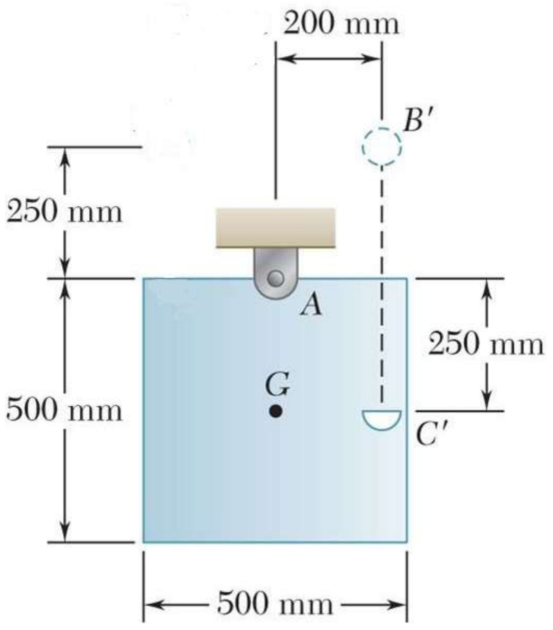
a) Apply linear-momentum to solve for mv_G

b) Use work-energy and set T_{final} equal to zero

c) Use sum of forces and sum of moments equations

Concept Question

For the previous problem, what would you do if you wanted to determine how high up the panel swung after the impact?



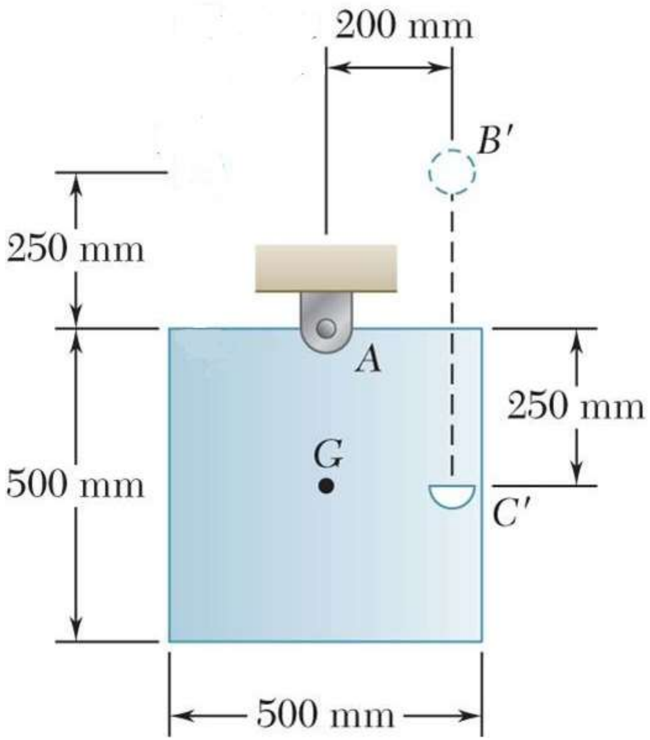
a) Apply linear-momentum to solve for mv_G

b) Use work-energy and set T_{final} equal to zero

c) Use sum of forces and sum of moments equations

Concept Question

For the previous problem, what if the ball was dropped closer to point A (example: at x= 100 mm instead of 200 mm)?



a) The angular velocity after impact would be bigger

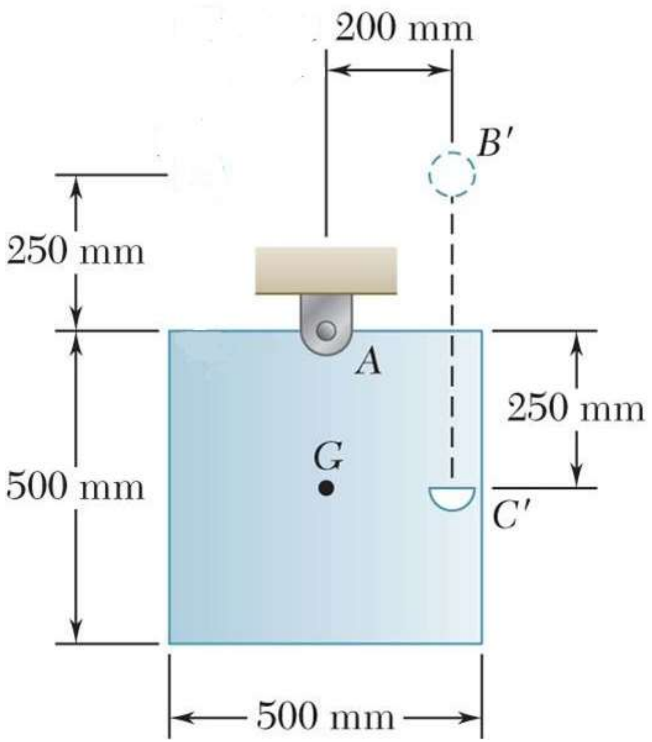
b) The angular velocity after impact would be smaller

c) The angular velocity after impact would be the same

d) Not enough information to tell

Concept Question

For the previous problem, what if the ball was dropped closer to point A (e.g., at x= 100 mm instead of 200 mm)?



a) The angular velocity after impact would be bigger

[✓ b) The angular velocity after impact would be smaller]

c) The angular velocity after impact would be the same

d) Not enough information to tell

End of Chapter 17