ASE2910 Applied Linear Algebra / AUS2910 Fundamental Math for AI
Homework #1

1) *Linear functions.*

   a) Show that an inner product function, $f(x) = a^T x$, is linear.

   b) Show that any scalar-valued linear function $f(x)$ satisfying superposition can be expressed as an inner product function, say $f(x) = a^T x$. Explicitly state the elements of a in terms of f.

2) *Affine functions.*

   a) Show that an inner product function plus a shift, $f(x) = a^T x + b$, is affine.

   b) Show that any scalar-valued affine function $f(x)$ satisfying the restricted superposition (superposition defined for linear combination with coefficients that sum to 1) can be expressed as an inner product function plus a shift, say $f(x) = a^T x + b$. Explicitly state the elements of a and b in terms of f.

3) *Delivery routing with two legs.* A courier in 3D first flies displacement $(2, 1, 3)$ and can optionally add any multiple of $(-1, 4, 0)$ using a booster path. Determine whether it is possible to reach $(7, 9, 6)$ by an appropriate linear combination. If so, find the coefficients.

4) *Mixing raw materials.* Each unit of compound A requires $(3, 1, 2)$ units of (Iron, Copper, Nickel) and compound B requires $(1, 2, 1)$. A lab has stock $(10, 7, 7)$. Can the stock be used exactly by producing (possibly fractional) amounts of A and B? If yes, find a solution; if not, explain why.

5) *Reachable positions on a grid.* A robot in $\mathbb{R}^2$ moves by repeating any combination of moves $m_1 = (4, 1)$ and $m_2 = (1, 3)$.

   a) Describe the set of all reachable points geometrically.

   b) Determine if $(19, 17)$ is reachable and give move counts if it is.

6) *Audio pattern synthesis.* Two audio clips: $s_1 = (1, 0, -1, 0)$, $s_2 = (0, 1, 0, -1)$. Target pattern $b = (3, 2, -3, -2)$. Can it be expressed as a linear combination of s_1, s_2? Find weights if yes.

7) *Cauchy–Schwarz in data correlation.* Centered data vectors $x, y \in \mathbb{R}^n$ have correlation

$$\rho = \frac{x^T y}{\|x\| \|y\|}.$$

Use Cauchy–Schwarz to show $|\rho| \leq 1$, and give a condition for equality.

8) *Parallelogram law in signal energy.* For signals $u, v \in \mathbb{R}^n$, define $\|w\|^2 = w^T w$. Prove $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$ and interpret.

9) *Nearest point on an affine line.* Let $L = \{tA + b : t \in \mathbb{R}\}$ with $A = (1, 2)$ and $b = (1, 0)$. For $p = (3, 1)$, find the point on L closest to p and the minimum distance.