ASE2910 Applied Linear Algebra / AUS2910 Fundamental Math for AI
Homework #2

1) *Matrix equality.* Let A and B be two $m \times n$ matrices. Under each of the assumptions below, determine whether $A = B$ must always hold, or whether $A = B$ holds only sometimes.

   a) Suppose $Ax = Bx$ holds for all n-vectors x.

   b) Suppose $Ax = Bx$ for some nonzero n-vector x.

2) *Orthogonal projection.* Let x be an n-vector and $u_1, \ldots, u_k$ with $k < n$ be orthonormal n-vectors. The projection of x onto the span of $u_1, \ldots, u_k$ is

$$\hat{x} = (u_1^T x)u_1 + \cdots + (u_k^T x)u_k \in \mathbb{R}^n$$

   and the projection of x onto the span of $u_1, \ldots, u_{k-1}$ is

$$\tilde{x} = (u_1^T x)u_1 + \cdots + (u_{k-1}^T x)u_{k-1} \in \mathbb{R}^n.$$

   Show that $\hat{x}$ is closer to x than $\tilde{x}$ in that

$$\|\hat{x} - x\| \leq \|\tilde{x} - x\|.$$

3) *Transformation by orthonormal vectors.* Suppose that the columns of $T \in \mathbb{R}^{n \times k}$ with $k \leq n$ are orthonormal. Show that the transformation $x \mapsto Tx$ satisfies the following properties.

   a) The transformation is *isometric*, *i.e.*, it preserves *distance* between vectors,

$$\|Tx - Ty\| = \|x - y\|, \quad \forall x, y.$$

   b) The transformation is *conformal*, *i.e.*, it preserves *angle* between vectors,

$$\angle(Tx, Ty) = \angle(x, y), \quad \forall x, y.$$

4) *Map on a Napkin.* You choose three non-orthogonal direction hints in $\mathbb{R}^3$,

$$u_1 = (1, 1, 0), \qquad u_2 = (0, 1, 1), \qquad u_3 = (1, 0, 1),$$

   and wish to construct an orthonormal world frame q_1, q_2, q_3 via Gram–Schmidt algorithm. Then, express the landmark

$$p = (6, 5, 4)$$

   in the q-coordinates, *i.e.*, compute $(\langle q_1, p\rangle,\ \langle q_2, p\rangle,\ \langle q_3, p\rangle)$. Provide all steps symbolically (with exact radicals), and verify orthonormality explicitly.

5) *VMLS Exercises.*

   a) **5.1** *Linear independence of stacked vectors.*

   b) **5.4** *Norm of linear combination of orthonormal vectors.*

   c) **5.5** *Orthogonalizing vectors.*

   d) **5.6** *Gram–Schmidt algorithm.*

   e) **5.8** *Early termination of Gram–Schmidt algorithm.*

   f) **6.3** *Block matrix.*

   g) **6.8** *Cash flow to bank account balance.*

   h) **6.10** *Resource requirements.*

   i) **6.12** *Skew-symmetric matrices.*

   j) **6.14** *Norm of matrix-vector product.*

   k) **6.17** *Stacked matrix.*