ASE2910 Applied Linear Algebra / AUS2910 Fundamental Math for AI
Homework #6

1) *Pseudoinverse commands for a redundant actuator system.* An idealized spacecraft actuator test rig uses three signed command signals to generate two normalized control-moment components. Its calibrated model is

$$f = Au, \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

where $u \in \mathbb{R}^3$ is the command vector and $f \in \mathbb{R}^2$ is the output. The command signals may take arbitrary real values, with no magnitude limits.

a) Show that the rows of A are linearly independent. Compute the pseudoinverse

$$A^\dagger = A^T(AA^T)^{-1},$$

and verify that $AA^\dagger = I$.

b) The desired output is

$$f_{\text{des}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Compute the command

$$u^* = A^\dagger f_{\text{des}}.$$

Then find *all* command vectors u satisfying $Au = f_{\text{des}}$, and identify u^* within this family.

c) An engineer measures the output $f = Au$ and proposes reconstructing the original command using

$$u_{\text{rec}} = A^\dagger f.$$

The engineer claims that $u_{\text{rec}} = u$ for every $u \in \mathbb{R}^3$.
Compute $P = A^\dagger A$ and determine whether the claim is correct. If it is false, give a nonzero counterexample. Then prove that

$$A(Pu) = Au \qquad \text{for every } u \in \mathbb{R}^3.$$

Explain the distinction between recovering the original command and reproducing its output.

2) *Estimating a drone's descent from noisy altitude measurements.* During a short descent interval, a drone's altitude is modeled as

$$h(t) = h_0 + vt,$$

where h_0 is the altitude at $t = 0$, in meters, and v is the constant signed vertical velocity, in meters per second.

The following altitude measurements are available:

Time t_i (s)	0	1	2	3
Measured altitude b_i (m)	10	9	9	6

Choose h_0 and v to minimize the sum of squared measurement residuals,

$$\sum_{i=1}^{4}(h_0 + vt_i - b_i)^2.$$

a) Express the problem as

$$\underset{x\in\mathbb{R}^2}{\text{minimize}} \quad \|Ax-b\|^2, \qquad x = \begin{bmatrix} h_0 \\ v \end{bmatrix}.$$

Specify A and b. Explain why no choice of h_0 and v fits all four measurements exactly.

b) Form the normal equations and compute the least-squares estimate

$$\hat{x} = \begin{bmatrix} \hat{h}_0 \\ \hat{v} \end{bmatrix}.$$

State whether the fitted model describes ascent or descent, and compute its predicted altitude at $t = 4$ s.

c) Compute the residual vector

$$r = A\hat{x} - b,$$

the sum of squared residuals $\|r\|^2$, and the RMS residual $\|r\|/\sqrt{4}$. Verify that $A^T r = 0$. What does this imply about the sum of the four residuals?

d) Without using derivatives, show that for every $x \in \mathbb{R}^2$,

$$\|Ax-b\|^2 = \|A\hat{x}-b\|^2 + \|A(x-\hat{x})\|^2.$$

Use this identity to explain why $\hat{x}$ is the *unique* least-squares solution.

3) *VMLS Exercises.*

a) **12.1** *Approximating a vector as a multiple of another one.*

b) **12.3** *Least angle property of least squares.*

c) **12.4** *Weighted least squares.*

d) **12.5** *Approximate right inverse.*

e) **12.6** *Least squares equalizer design.*

f) **12.8** *Least squares and QR factorization.*

g) **12.13** *Iterative method for least squares problem.*

h) **12.15** *Minimizing a squared norm plus an affine function.*