9. Linear dynamical systems

Outline

Linear dynamical systems

Population dynamics

Epidemic dynamics

State sequence

- sequence of n-vectors $x_1, x_2, \ldots$
- t denotes time or period
- x_t is called *state* at time t; sequence is called *state trajectory*
- assuming t is current time,
  - x_t is current state
  - x_{t-1} is previous state
  - x_{t+1} is next state
- examples: x_t represents
  - age distribution in a population
  - economic output in n sectors
  - mechanical variables

Linear dynamics

- linear dynamical system:

$$x_{t+1} = A_t x_t, \quad t = 1, 2, \ldots$$

- A_t are $n \times n$ dynamics matrices
- $(A_t)_{ij}(x_t)_j$ is contribution to $(x_{t+1})_i$ from $(x_t)_j$
- system is called *time-invariant* if $A_t = A$ doesn't depend on time
- can simulate evolution of x_t using recursion $x_{t+1} = A_t x_t$

Variations

- linear dynamical system with input

$$x_{t+1} = A_t x_t + B_t u_t + c_t, \quad t = 1, 2, \ldots$$

  - u_t is an *input* m-vector
  - B_t is $n \times m$ *input matrix*
  - c_t is *offset*

- K-Markov model:

$$x_{t+1} = A_1 x_t + \cdots + A_K x_{t-K+1}, \quad t = K, K+1, \ldots$$

  - next state depends on current state and $K - 1$ previous states
  - also known as *auto-regresssive model*
  - for $K = 1$, this is the standard linear dynamical system $x_{t+1} = Ax_t$

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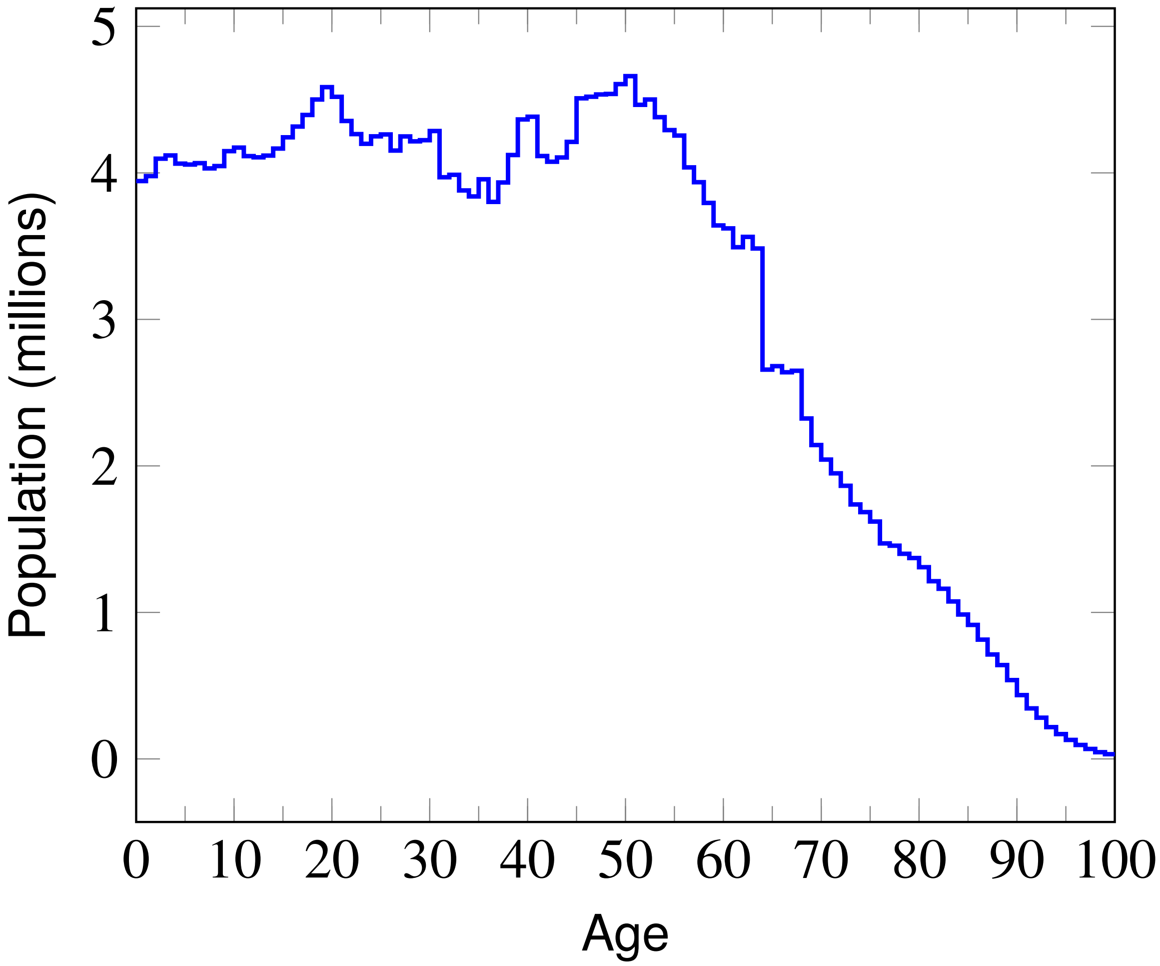
Population distribution

- $x_t \in \mathbf{R}^{100}$ gives population distribution in year $t = 1, \ldots, T$
- $(x_t)_i$ is the number of people with age $i - 1$ in year t (say, on January 1)
- total population in year t is $\mathbf{1}^T x_t$
- number of people age 70 or older in year t is $(0_{70}, \mathbf{1}_{30})^T x_t$

Population distribution of the U.S.

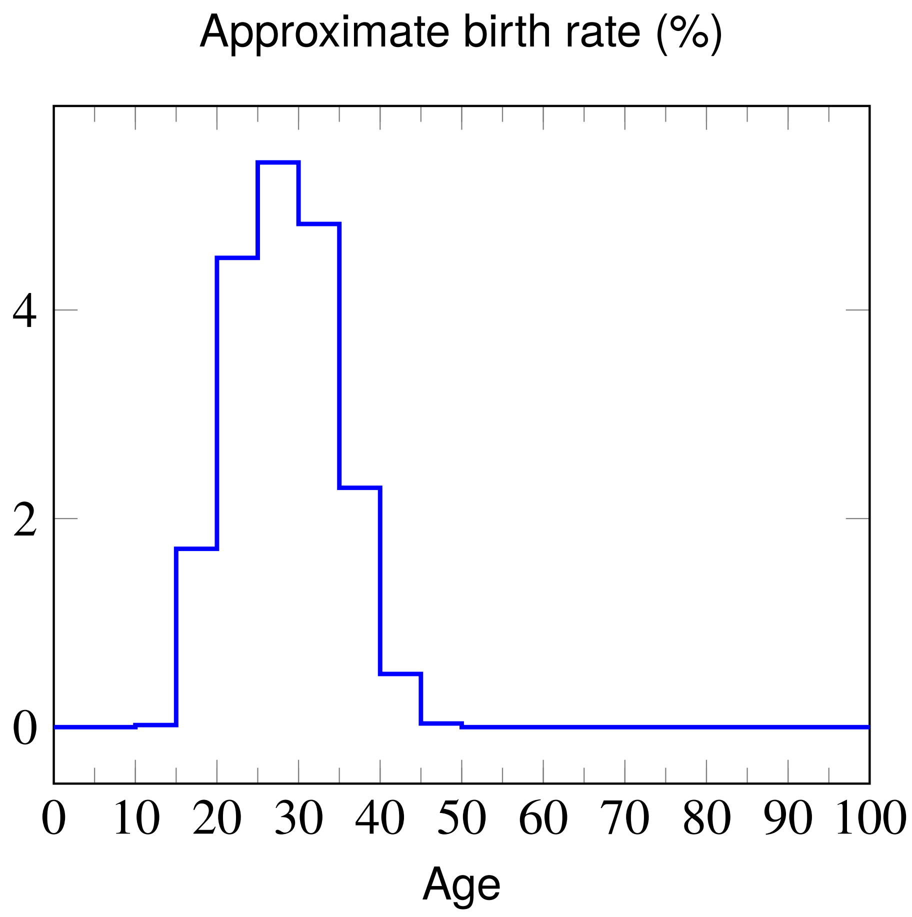
(from 2010 census)

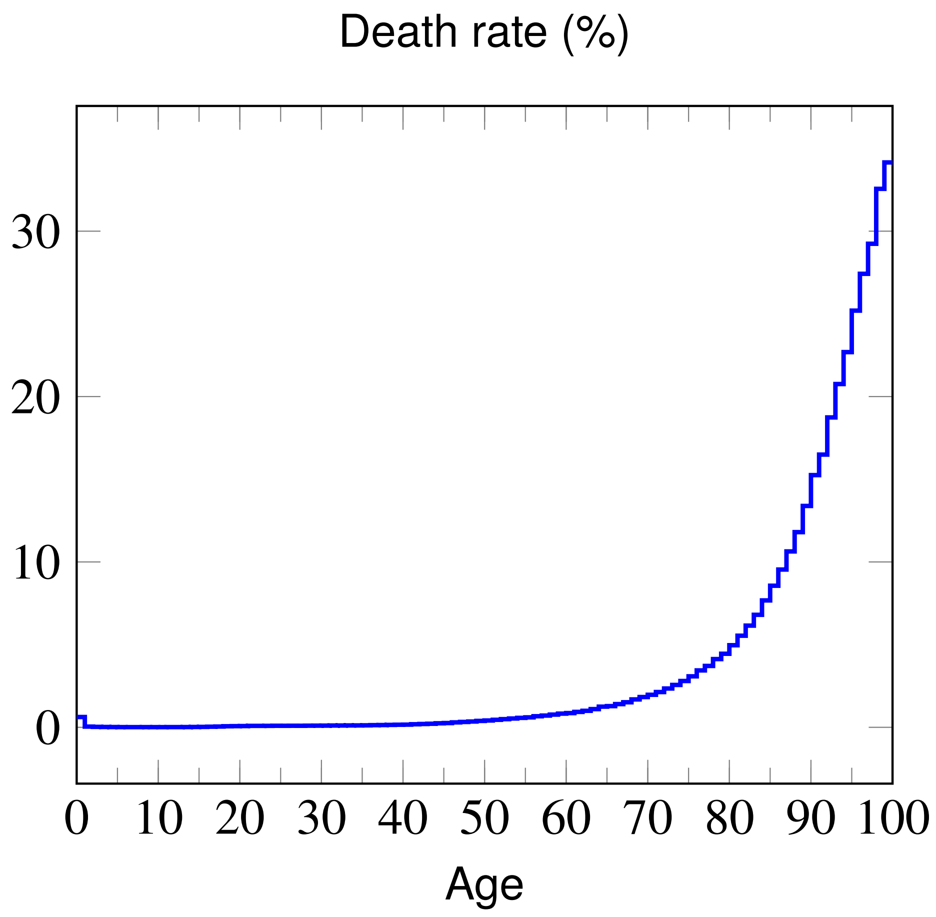


Birth and death rates

- birth rate $b \in \mathbf{R}^{100}$, death (or mortality) rate $d \in \mathbf{R}^{100}$
- b_i is the number of births per person with age $i-1$
- d_i is the portion of those aged $i-1$ who will die this year (we'll take $d_{100} = 1$)
- b and d can vary with time, but we'll assume they are constant

Birth and death rates in the U.S.





Dynamics

- let's find next year's population distribution x_{t+1} (ignoring immigration)
- number of 0-year-olds next year is total births this year:

$$(x_{t+1})_1 = b^T x_t$$

- number of i-year-olds next year is number of $(i-1)$-year-olds this year, minus those who die:

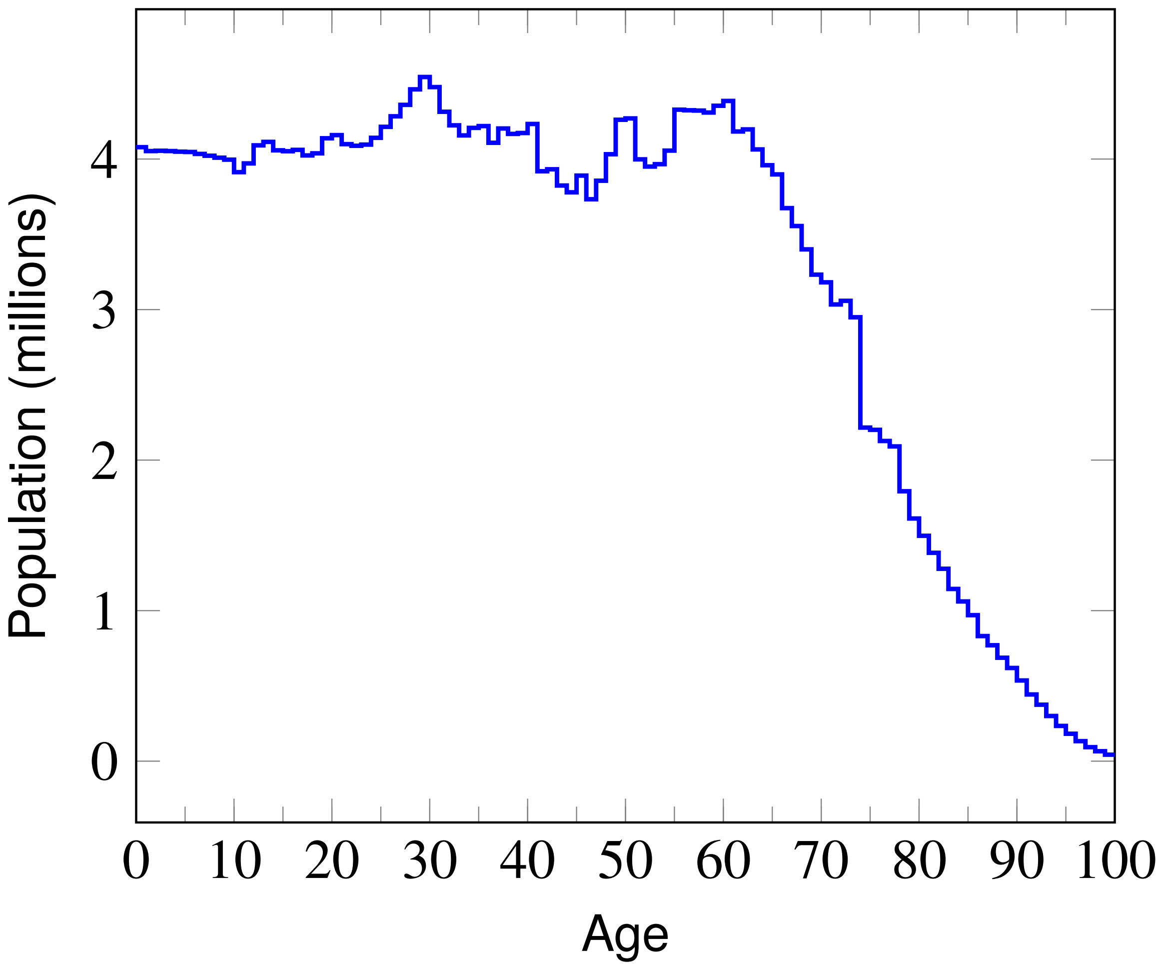
$$(x_{t+1})_{i+1} = (1-d_i)(x_t)_i, \quad i = 1, \ldots, 99$$

- $x_{t+1} = Ax_t$, where

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 0 \end{bmatrix}$$

Predicting future population distributions

predicting U.S. 2020 distribution from 2010 (ignoring immigration)



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SIR model

- 4-vector x_t gives proportion of population in 4 infection states

Susceptible:	can acquire the disease the next day
Infected:	have the disease
Recovered:	had the disease, recovered, now immune
Deceased:	had the disease, and unfortunately died

- sometimes called *SIR model*
- *e.g.*, $x_t = (0.75, 0.10, 0.10, 0.05)$

Epidemic dynamics

over each day,

- among susceptible population,
  - 5% acquires the disease
  - 95% remain susceptible
- among infected population,
  - 1% dies
  - 10% recovers with immunity
  - 4% recover without immunity (*i.e.*, become susceptible)
  - 85% remain infected
- 100% of immune and dead people remain in their state
- epidemic dynamics as linear dynamical system

$$x_{t+1} = \begin{bmatrix} 0.95 & 0.04 & 0 & 0 \\ 0.05 & 0.85 & 0 & 0 \\ 0 & 0.10 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{bmatrix} x_t$$

Simulation from $x_1 = (1,0,0,0)$