ASE3093 Automatic Control: Homework #2

1) *Linearity of the Laplace transform.* Let the Laplace transform of a time-domain function $f(t)$ be defined as:

$$\mathcal{L}\{f(t)\} = F(s)$$

   a) Show that $\mathcal{L}\{\alpha f(t)\} = \alpha F(s)$ for any constant $\alpha \in \mathbb{R}$.

   b) Show that $\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$.

2) *Convolution-to-product property of the Laplace transform.* Let a linear-time-invariant (LTI) system have impulse response $h(t)$, and let $u(t)$ be the system input. Then the output $y(t)$ is given by the convolution:

$$\begin{aligned} y(t) &= (h * u)(t) \\ &= \int_0^t u(\tau)h(t-\tau)\,d\tau \\ &= \int_0^t h(\tau)u(t-\tau)\,d\tau \end{aligned}$$

Let $\mathcal{L}\{f(t)\} = F(s)$ denote the Laplace transform of a function $f(t)$, defined as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Using the definitions above, prove that:

$$\begin{aligned} \mathcal{L}\{y(t)\} &= \mathcal{L}\{h(t) * u(t)\} \\ &= H(s) \cdot U(s) \end{aligned}$$

That is, prove that the Laplace transform of a convolution is the product of the Laplace transforms.

3) *System identification.* Consider a first-order system described by the following dynamics:

$$\dot{y}(t) = ay(t) + bu(t)$$

The system is initially at rest. A unit step input $u_s(t)$ is applied, and the output is measured at two time points: after 0.5 seconds, and after a sufficiently long time. The measured values are:

$$y(0.5) = 1, \quad y(\infty) = 2$$

Using the measurements above, determine the system parameters a and b.

4) *Transfer functions.* Find the transfer function of the system, $G(s) = \frac{Y(s)}{U(s)}$, from the each of the following linear systems.

a) The system follows the following differential equation:

$$6\ddot{y}(t) + 11\dot{y}(t) + 6y(t) = u(t)$$

b) The system has the following impulse response:

$$h(t) = 4e^{-t} - 2te^{-t}$$

for $t \geq 0$ and $h(t) = 0$ otherwise.

c) The system has the following step response:

$$y(t) = 1 - e^{-t}\cos t - e^{-t}\sin t$$

for $t \geq 0$ and $h(t) = 0$ otherwise.

d) The system follows the following differential equations:

$$\begin{aligned} &\ddot{y}(t) + 2\dot{y}(t) + y(t) = \dot{x}(t) + x(t) \\ &\dot{x}(t) + 2x(t) - 2y(t) = u(t) \end{aligned}$$

5) *Drawing exercise.* Given the transfer function $G(s)$ of a linear system:

$$G(s) = \frac{s^2 - 26}{(s+10)(s^2+3s+4)}$$

Answer the following questions:

a) Express the above transfer function as the sum of two simpler rational functions using partial fraction decomposition.

b) Without using a computer, sketch the step response of each component, and then sketch the step response of $G(s)$ as a linear combination of the individual responses.

c) What is the steady-state value of $y(t)$, that is $\lim_{t\to\infty} y(t)$, exactly?