ASE3093 Automatic Control / AUS3204 Applied Control Engineering
Homework #5

1) *First-order system frequency response.* Consider the first-order transfer function

$$G(s) = \frac{1}{\tau s + 1}.$$

Suppose the input is a sine wave $r(t) = X\sin(\omega t)$.

a) Substitute $s = j\omega$ into $G(s)$ to obtain $G(j\omega)$, and rationalize the resulting complex fraction so that it has the form $R + jI$. Express R and I in terms of ω and τ.

b) Show that the steady-state output is

$$c_{\text{ss}}(t) = X|G(j\omega)|\sin(\omega t + \phi),$$

where $|G(j\omega)| = 1/\sqrt{1 + (\omega\tau)^2}$ and the phase shift is $\phi = -\arctan(\omega\tau)$.

c) Sketch qualitatively how the amplitude ratio $|G(j\omega)|$ and phase ϕ vary with frequency. Explain why the magnitude is close to 1 for $\omega \ll 1/\tau$ and decreases as ω increases, while the phase lag approaches $-90°$.

2) *Identifying cutoff frequency.* Consider the transfer function $G(s) = \frac{1}{\tau s+1}$ from the previous problem. The *cutoff frequency* (or break frequency) ω_c is defined as the frequency at which the magnitude $|G(j\omega)|$ drops to $1/\sqrt{2}$ of its low-frequency value.

a) Show that $\omega_c = 1/\tau$ for this system.

b) If $\tau = 0.2$ s, estimate the amplitude ratio $|G(j\omega)|$ at $\omega = 0.1\,\omega_c$, $\omega = \omega_c$ and $\omega = 10\,\omega_c$. Comment on how the system behaves as a low-pass filter.

3) *Comparing two first-order systems by their frequency response.* Consider two systems

$$G_1(s) = \frac{1}{0.2s + 1}, \qquad G_2(s) = \frac{1}{2s + 1}.$$

a) For each system, compute the magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ at $\omega = 1$ rad/s.

b) Which system has the larger phase lag at this frequency? Which one attenuates the input more?

c) Briefly explain how differences in τ affect the frequency response shape.

4) *Complementary filter in the frequency domain.* Suppose two sensors, $x_1(t)$ and $x_2(t)$, are both measuring the same underlying physical signal (e.g., an orientation angle), but with different error characteristics across the frequency spectrum:

- $x_1(t)$ is obtained from an accelerometer — accurate at low frequencies but dominated by high-frequency noise,
- $x_2(t)$ is obtained from a gyroscope — reliable at high frequencies but subject to low-frequency drift.

To fuse these signals in a frequency-sensitive manner, a *complementary filter* is used. The estimated signal $\hat{x}(s)$ is formed as:

$$\hat{x}(s) = G_{\text{LP}}(s)x_1(s) + G_{\text{HP}}(s)x_2(s)$$

where:

$$G_{\text{LP}}(s) = \frac{1}{\tau s + 1}, \qquad G_{\text{HP}}(s) = \frac{\tau s}{\tau s + 1}$$

These filters are complementary, satisfying:

$$G_{\text{LP}}(s) + G_{\text{HP}}(s) = 1$$

a) Show that $G_{\text{LP}}(s) + G_{\text{HP}}(s) = 1$ for all s.

b) Sketch or plot the Bode magnitude plots of $G_{\text{LP}}(s)$ and $G_{\text{HP}}(s)$ for $\tau = 0.1$.

c) Describe the behavior of both filters in the following frequency regimes:
   - Low frequencies ($\omega \ll \frac{1}{\tau}$),
   - High frequencies ($\omega \gg \frac{1}{\tau}$),
   - Crossover frequency ($\omega = \frac{1}{\tau}$).

d) Explain how the complementary filter leverages the strengths of both x_1 and x_2 to provide a more robust and accurate estimate $\hat{x}(t)$ across the entire frequency range.

5) *Bode plots.* For each of the following systems, sketch the Bode magnitude and phase plots by hand. Then verify your results using a computer-based tool (e.g., Python or MATLAB).

a) $$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

b) $$G(s) = \frac{1000(s+1)}{s(s+2)(s^2+8s+64)}$$

c) $$G(s) = \frac{4s(s+10)}{(s+50)(4s^2+5s+4)}$$

d) $$G(s) = \frac{s+2}{s^2(s+20)}$$

e) $$G(s) = \frac{(s+0.5)(s+1.5)}{s(s^2+2s+2)(s+5)(s+15)}$$

f) $$G(s) = \frac{s+1}{(s+2)(s+10)}$$

g) $$G(s) = \frac{s-1}{(s-2)(s+10)}$$