ASE6029 Linear Optimal Control
Homework #1

1) *Diagonalization.* Show that a matrix with distinct eigenvalues is diagonalizable.

2) *Symmetric matrices.*

   a) Show that a symmetric matrix has real eigenvalues.

   b) Show that a symmetric matrix with distinct eigenvalues is orthogonally diagonalizable.

   c) Show that a symmetric matrix is orthogonally diagonalizable.

   d) Say the eigenvalues of $A \in \mathbb{S}^n$ are ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Show that

   $$\lambda_n \|x\|^2 \leq x^T A x \leq \lambda_1 \|x\|^2$$

   and explain when the inequalities are tight.

3) *Range and Kernel of a symmetric matrix.* Show that the range (image) and the kernel (null space) of A are disjoint and orthonal. Furthermore the union of them form $\mathbb{R}^n$.

4) *Simultaneous diagonalizability.* Two matrices A and B are said to be simultaneously diagonalizable if there exists an invertible matrix T such that both $T^{-1}AT$ and $T^{-1}BT$ are diagonal. Now supppose that A and B are simultaneously diagonalizable.

   a) Show that they commute:

   $$AB = BA$$

   b) Show that the product rule for matrix exponentials holds in this case..

   $$e^{A+B} = e^A e^B$$

5) *Stability of a switched linear system with Hurwitz modes.* Consider the time-varying linear system

$$\dot{x}(t) = A(t)\,x(t), \qquad A(t) = A_{\sigma(t)},$$

where $\sigma : [0,\infty) \to \mathbb{N}$ is a piecewise-constant switching signal with switching times $0 < t_1 < t_2 < \cdots$, so that

$$A(t) = \begin{cases} A_0, & 0 \le t < t_1, \\ A_1, & t_1 \le t < t_2, \\ \vdots \end{cases}$$

Assume that each mode matrix $A_i \in \mathbb{R}^{n\times n}$ is Hurwitz (all eigenvalues lie in the open left half–plane). Is the switched system necessarily (globally) asymptotically stable under *arbitrary* switching, *i.e.*, does $x(t) \to 0$ as $t \to \infty$ for every initial condition and every switching signal?

6) *Complex eigenvectors.* Consider a linear dynamical system $\dot{x} = Ax$, where $Av = \lambda v$, with $v \neq 0$ and complex $\lambda \in \mathbb{C}$.

a) Show that the complex trajectory

$$x(t) = ae^{\lambda t}v,$$

for $a \in \mathbb{C}$ satisfies $\dot{x} = Ax$.

b) Show that the following real trajectory

$$x(t) = e^{\sigma t}\begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where $v = v_{\text{re}} + jv_{\text{im}}$, $\lambda = \sigma + j\omega$, and $a = \alpha - j\beta$.