ASE6029 Linear Optimal Control
Homework #2

1) *A system with mixed eigenvalues.* Consider a linear dynamical system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$ where A is diagonalizable and has mixed eigenvalues such that

$$\Re\lambda_1 < 0, \ldots, \Re\lambda_s < 0,$$

for some $s < n$ and

$$\Re\lambda_{s+1} \geq 0, \ldots, \Re\lambda_n \geq 0.$$

Let v_i and w_i be the (right) eigenvector and the left eigenvector associated with the i-th eigenvalue, λ_i, that is,

$$Av_i = \lambda_i v_i \qquad \text{and} \qquad w_i^T A = \lambda_i w_i^T.$$

a) Show that $x(t)$ for an arbitrary initial state $x(0)$ is given by:

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T x(0)$$

b) Show that $x(t) \to 0$ as $t \to \infty$ if

$$w_i^T x(0) = 0, \qquad \text{for } i = s+1, \ldots, n.$$

c) Show that the above condition is equivalent to the following.

$$x(0) \in \textbf{span}\{v_1, \ldots, v_s\}.$$

In other words, $x(t) \to 0$ as $t \to \infty$ in this case.

2) *Solution to linear dynamical systems with inputs and outputs.* Consider the linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $u : [0, T] \to \mathbb{R}^m$ is continuous.

Then the state trajectory, $x(t)$ for $0 \leq t \leq T$, is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, d\tau.$$

Explain why.

3) *Shift matrix.* Consider the $n \times n$ upper shift matrix,

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with

$$J = \lambda I + N = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

a) Find e^{tN}.

b) Find e^{tJ}. *Hint: Note that λI and N commute.*

4) *Limit behaviours of the regularized least squares solutions.* In this problem we consider the optimal solution of the Tykhonov regularized least squares solutions with a full rank matrix $X \in \mathbb{R}^{n\times d}$.

$$\begin{aligned} \theta^* &= \arg\min_{\theta} ||X\theta - y||_2^2 + \lambda||\theta||_2^2 \\ &= \left(X^T X + \lambda I\right)^{-1} X^T y \end{aligned}$$

Suppose X is skinny and full rank, *i.e.*, $\text{rank}(A) = d < n$. Then it is crystal clear that $\theta^* \to 0$ as $\lambda \to \infty$, and $\theta^* \to (X^T X)^{-1} X^T y$ as $\lambda \to 0$, that is, θ^* approaches to zero if λ is extremely large, and θ^* approaches to the unregularized least squares solution when λ is tiny. No big deal.

Now consider the opposite case when X is fat and full rank, *i.e.*, $\text{rank}(A) = n < d$. The optimal θ^* is zero for extremely large λ. The same thing. However an interesting thing happens when λ approaches to zero. In this problem, we are going to look at that. First note that $X^T X$, which is the limit of $(X^T X + \lambda I)$ as $\lambda \to 0$, is not invertible in this case, hence the expression $(X^T X)^{-1} X^T y$ doesn't make sense at all.

a) Show that the following holds whenever the appearing matrix products and inverses make sense. It is called the *Push-through identity.*

$$A\left(I + BA\right)^{-1} = \left(I + AB\right)^{-1} A$$

b) Find the optimal θ^* by applying the above to your optimal regularized least squares solution, and taking the limit, $\lambda \to 0$. What is it?

c) Show that your solution satisfies $X\theta^* = y$.

d) Show that $||\theta^*||_2 \leq ||\theta||_2$ for any $\theta \in \mathbb{R}^d$.

The solution you found achieves the minimum norm among the infinitely many solutions satisfying $X\theta = y$, hence it is called the *least norm* solution.