ASE6029 Linear Optimal Control
Homework #3

1) *LQR Exercises.* Consider the following finite-horizon discrete time LQR problem.

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{t=0}^{N-1} \left(x_t^T Q x_t + u_t^T R u_t\right) + x_N^T Q_f x_N \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 0, \dots, N-1, \end{array}$$

where the parameters are:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$Q = C^\top C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_f = Q, \quad R = \rho I,$$

and the system begins from $x_0 = (1, 0)$.

Solve the problem via the dynamic-programming (Riccati) recursion:

$$\begin{aligned} P_N &= Q_f, \\ P_t &= Q + A^T P_{t+1} A - A^T P_{t+1} B \left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A, \\ K_t &= -\left(R + B^T P_{t+1} B\right)^{-1} B^T P_{t+1} A, \qquad u_t = K_t x_t, \end{aligned}$$

for $t = N-1, \dots, 0$, and compute the performance terms

$$J_{\text{in}} = \sum_{t=0}^{N-1} \|u_t\|_2^2, \qquad J_{\text{out}} = \sum_{t=0}^{N} \|y_t\|_2^2.$$

a) *Trade-off curve:* Sweep ρ over a dense grid (e.g., 100–200 values on a log scale) and plot J_{out} versus J_{in} as a smooth curve.

b) *Input and output sequences:* For $\rho = 0.3$ and $\rho = 10$, simulate the closed loop and make two stacked plots sharing the t-axis: top panel u_t vs. t; bottom panel y_t vs. t.

c) *State-feedback gains:* Plot $(K_t)_1$ and $(K_t)_2$ versus t for three terminal weights: $Q_f = Q$, $Q_f = \mathbf{0}$, and $Q_f = 10^3 I$.

d) *Time evolution of P_t:* In addition to computing the control gains K_t, extract the Riccati matrices P_t at each time t. Plot the elements of $P_t \in \mathbb{R}^{2\times 2}$ as functions of t for $t = 0, \dots, N$. That is, generate four plots for:

$$(P_t)_{11}, \quad (P_t)_{12}, \quad (P_t)_{21}, \quad (P_t)_{22}.$$

Present them as subplots (4-by-1 grid). Use $\rho = 1$.

e) *Finite-horizon vs. steady-state LQR:* Compare the finite-horizon LQR solution (as computed above) with the steady-state (infinite-horizon) LQR solution.

   i) Compute the steady-state solution P_∞ by iterating the Riccati equation until convergence.

   ii) Extract the steady-state gain $K_\infty = -(R + B^\top P_\infty B)^{-1} B^\top P_\infty A$.

   iii) Simulate both systems from the same initial condition $x_0 = (1, 0)$ under:

$$u_t^{\text{finite}} = K_t x_t, \qquad u_t^{\text{steady}} = K_\infty x_t.$$

   iv) Plot the trajectories of x_t, u_t, and y_t for both policies over time ($t = 0, \ldots, N$). Provide plots comparing their control and output behavior.