ASE6029 Linear optimal control: Homework #2

1) *A system with mixed eigenvalues.* Consider a linear dynamical system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$ where A is diagonalizable and has mixed eigenvalues such that

$$\Re\lambda_1 < 0, \ldots, \Re\lambda_s < 0,$$

for some $s < n$ and

$$\Re\lambda_{s+1} \geq 0, \ldots, \Re\lambda_n \geq 0.$$

Let v_i and w_i be the (right) eigenvector and the left eigenvector associated with the i-th eigenvalue, λ_i, that is,

$$Av_i = \lambda_i v_i \qquad \text{and} \qquad w_i^T A = \lambda_i w_i^T.$$

a) Show that $x(t)$ for an arbitrary initial state $x(0)$ is given by:

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T x(0)$$

b) Show that $x(t) \to 0$ as $t \to \infty$ if

$$w_i^T x(0) = 0, \qquad \text{for } i = s+1, \ldots, n.$$

c) Show that the above condition is equivalent to the following.

$$x(0) \in \mathbf{span}\{v_1, \ldots, v_s\}.$$

In other words, $x(t) \to 0$ as $t \to \infty$ in this case.

2) Shift matrix. Consider the $n \times n$ upper shift matrix,

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with

$$J = \lambda I + N = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

a) Find e^{tN}.

b) Find e^{tJ}. *Hint: Note that λI and N commute.*